A4.1 Let $P_2$ be the vector space of real polynomials in $t$ of degree at most 2 and let $T_1 : \mathbb{R}^3 \to P_2$ and $T_2 : P_2 \to \mathbb{R}^3$ be the linear transformations given by

\[ T_1(x_1, x_2, x_3) = x_1 + x_2t + x_3t^2 \]

and

\[ T_2(f(t)) = (2f(0) - f(1), 3f(1) - 2f(2), 4f(2) - 3f(3)). \]

(a) Find the kernels and ranges of $T_1$ and $T_2$.

(b) Find $T_1 \circ T_2$ and $T_2 \circ T_1$.

(c) Let $B_1 = \{e_1, e_2, e_3\}$ and $B_2 = \{1, t, t^2\}$ be the standard bases of $\mathbb{R}^3$ and $P_2$, respectively. Compute $[T_1]_{B_2 \leftarrow B_1}$, $[T_2]_{B_2 \leftarrow B_1}$, $[T_1 \circ T_2]_{B_2 \leftarrow B_1}$ and $[T_2 \circ T_1]_{B_1 \leftarrow B_1}$.

A4.2 Let $L(V,W)$ be the vector space of all linear transformations between two finite-dimensional vector spaces $V$ and $W$ and let $f : L(V,W) \to L(V,W)$ be a linear endomorphism on $L(V,W)$. Show that if $\text{rank}(T) \leq \text{rank}(f(T))$ for all $T \in L(V,W)$, then $f$ is invertible.

A4.3 Let $T_1 : V \to W$ and $T_2 : V \to W$ be two linear transformations between two vector spaces $V$ and $W$. Prove the following:

(a) $K(T_1) \cap K(T_2) \subset K(T_1 + T_2)$.

(b) If $T_1(V) \cap T_2(V) = \{0\}$, then $K(T_1 + T_2) = K(T_1) \cap K(T_2)$.

A4.4 We call a linear endomorphism $P : V \to V$ a projection if $P^2 = P$. Show that $K(P) \cap P(V) = \{0\}$ for all projections $P : V \to V$.

A4.5 Let $T : V \to V$ be a linear endomorphism on a vector space $V$ of $\text{dim} \, V = n$. Suppose that $T^N = 0$ for some positive integer $N$. Show that $T^n = 0$. 