A3.1 For a subset $U$ of $V$, the inclusion $i_U : U \to V$ is the map $i_U(u) = u$ for $u \in U$. Let $V$ be a vector space and $U$ be a subspace of $V$. Do the following:

(a) Show that $i_U$ is an injective linear transformation.
(b) Let $T : V \to W$ be a linear transformation between vector spaces $V$ and $W$. Show that $K(T \circ i_U) = K(T) \cap U$ and $\dim(K(T) \cap U) = \dim U - \dim T(U)$.

A3.2 The inverse image of $S \subset W$ under a map $T : V \to W$ is

$$T^{-1}(S) = \{v \in V : T(v) \in S\}.$$  

Let $T : V \to W$ be a linear transformation between two vector spaces $V$ and $W$. Do the following:

(a) Show that $T^{-1}(W_0)$ is a subspace of $V$ and $K(T) \subset T^{-1}(W_0)$ for every subspace $W_0$ of $W$.
(b) Suppose that $\dim V < \infty$. Show that $\dim T^{-1}(W_0) = \dim(W_0 \cap T(V)) + \dim K(T)$ for all subspaces $W_0$ of $W$.

A3.3 Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces and let $B = \{v_1, v_2, ..., v_n\}$ be a basis of $V$. Do the following:

(a) Show that $T$ is 1-1 if and only if $T(v_1), T(v_2), ..., T(v_n)$ are linearly independent in $W$.
(b) Show that $T$ is onto if and only if $W = \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\}$.
(c) Show that $T$ is invertible if and only if $\{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis of $W$.

A3.4 Let $P \in M_{3 \times 3}(\mathbb{R})$ and $T : M_{3 \times 3}(\mathbb{R}) \to M_{3 \times 3}(\mathbb{R})$ be the map given by $T(A) = PA$.

(a) Show that $T$ is a linear transformation.
(b) Show that $T$ is invertible if and only if $P$ is an invertible matrix.
(c) Suppose that $\text{rank}(P) = 2$. Find $\text{rank}(T)$. You must justify your answer.
A3.5 Do the following:

(a) Let $T : U \rightarrow W$ be a linear transformation between two vector spaces $U$ and $W$. Show that $\text{rank}(T) \leq 1$ if and only if there exist a vector space $V$ of $\text{dim} V = 1$ and two linear transformations $T_1 : V \rightarrow W$ and $T_2 : U \rightarrow V$ such that $T = T_1 \circ T_2$.

(b) Let $A$ be an $m \times n$ matrix. Show that $\text{rank}(A) \leq 1$ if and only if there exist an $m \times 1$ matrix $A_1$ and a $1 \times n$ matrix $A_2$ such that $A = A_1A_2$. 