(1) No books, notes or calculators are allowed.
(2) Show your work in details.

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(1) (60 pts) Which of the following statements are true and which are false? Justify your answer.

(a) (10 pts) For a linear endomorphism \( T : V \rightarrow V \) on a vector space \( V \), if \( R(T) = R(T^2) \), then \( R(T^{2014}) = R(T^{2015}) \), where \( R(T) \) is the range of \( T \).

(b) (10 pts) For every \( n \times n \) matrix \( A \) and \( \lambda \in \mathbb{C} \),
\[
\text{Nul}(A - \lambda I)^n = \text{Nul}(A - \lambda I)^{n+1}.
\]
(c) (10 pts) For $n \times n$ matrices $A, B, C, D$, if $A$ is similar to $C$ and $B$ is similar to $D$, then $A + B$ must be similar to $C + D$.

(d) (10 pts) For every linear endomorphism $T : V \rightarrow V$ on a vector space $V$, a subspace $W$ of $V$ is $T$-invariant if and only if $W$ is $T^2$-invariant.
(e) (10 pts) For every linear endomorphism $T : V \to V$ on a finite-dimensional vector spaces $V$, 
\[ K(T^4 - I) = K(T - I) + K(T + I) + K(T^2 + I). \]

(f) (10 pts) Two square matrices $A$ and $B$ are similar if and only if they have the same characteristic and minimal polynomials.
(2) (15 pts) Solve the following system of ordinary differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 + x_2 \\
\frac{dx_2}{dt} &= x_1 + 2x_2 - x_3 \\
\frac{dx_3}{dt} &= x_1 + 2x_2
\end{align*}
\]

with

\[
\begin{align*}
x_1(0) &= 1 \\
x_2(0) &= 2 \\
x_3(0) &= 3
\end{align*}
\]
(3) (40 pts) Let $A$ be a square matrix satisfying
\[
\{\dim \text{Nul}(A - 2I)^k : k = 0, 1, \ldots\} = \{0, 2, 3, 3\ldots\}
\]
\[
\{\dim \text{Nul}(A + I)^k : k = 0, 1, \ldots\} = \{0, 4, 5, 6, 6, \ldots\}
\]
\[\dim \text{Nul}(A - \lambda I)^k = 0 \text{ for all } \lambda \neq -1, 2 \in \mathbb{C} \text{ and all } k \in \mathbb{N}.\]

(a) (10 pts) Find the minimal and characteristic polynomials of $A$.

(b) (10 pts) Find the Jordan Canonical Form of $A$. 
(c) (10 pts) Find the minimal and characteristic polynomials of $A - 2A^{-1}$. Justify your answer.
(d) (10 pts) Find the Jordan Canonical Form of $A - 2A^{-1}$. Justify your answer.
(4) (35 pts) Let $M_{m \times n}(\mathbb{R})$ be the vector space of $m \times n$ real matrices and $T : M_{2 \times 3}(\mathbb{R}) \to M_{2 \times 3}(\mathbb{R})$ be the linear endomorphism given by

$$T(A) = DA$$

where $D = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(a) (20 pts) Find the minimal and characteristic polynomials of $T$. Is $T$ diagonalizable? Justify your answer.
(b) (15 pts) Find $T_{2015}(A)$ for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$
(5) (25 pts) Let $A$ be an $n \times n$ matrix satisfying that $A^3 = 0$.

(a) (15 pts) Show that

$$\text{rank}(A) \leq \frac{2n}{3}$$
(b) (10 pts) Find a $6 \times 6$ matrix $A$ satisfying that $A^3 = 0$ and $\text{rank}(A) = 4$. Justify your answer.
(6) (25 pts) A complex square matrix $A$ is \textit{unipotent} if $(A - I)^m = 0$ for some positive integer $m$.

(a) (15 pts) Show that $A$ is unipotent if and only if 1 is the only eigenvalue of $A$.

(b) (10 pts) Show that $A$ and $A^{-1}$ are similar for all unipotent matrices $A$. 
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