Solutions for Math 325 Final

(1) Which of the following statements are true and which are false? Justify your answer.

(a) For a linear endomorphism \( T : V \to V \) on a vector space \( V \), if \( R(T) = R(T^2) \), then \( R(T^{2014}) = R(T^{2015}) \), where \( R(T) \) is the range of \( T \).

Proof. True since
\[
R(T) = R(T^2) \implies T(V) = T^2(V) \\
\implies T^{2013}(V) = T^{2013}(T^2(V)) \\
\implies T^{2014}(V) = T^{2015}(V) \\
\implies R(T^{2014}) = R(T^{2015}).
\]

(b) For every \( n \times n \) matrix \( A \) and \( \lambda \in \mathbb{C} \),
\[
\text{Nul}(A - \lambda I)^n = \text{Nul}(A - \lambda I)^{n+1}.
\]

Proof. True. Since the sequence \( \dim \text{Nul}(A - \lambda I)^k \) stabilizes, we have
\[
0 < \dim \text{Nul}(A - \lambda I) < \ldots < \dim \text{Nul}(A - \lambda I)^m = \dim \text{Nul}(A - \lambda I)^{m+1} = \ldots \leq n
\]
for some \( m \in \mathbb{N} \). Therefore,
\[
n \geq \dim \text{Nul}(A - \lambda I)^m = (\dim \text{Nul}(A - \lambda I) - 0) + (\dim \text{Nul}(A - \lambda I)^2 - \dim \text{Nul}(A - \lambda I)) + \ldots + (\dim \text{Nul}(A - \lambda I)^m - \dim \text{Nul}(A - \lambda I)^{m-1}) \geq m
\]
\[
\implies \dim \text{Nul}(A - \lambda I)^n = \dim \text{Nul}(A - \lambda I)^{n+1}.
\]
And since \( \text{Nul}(A - \lambda I)^n \subset \text{Nul}(A - \lambda I)^{n+1} \), we conclude
\[
\text{Nul}(A - \lambda I)^n = \text{Nul}(A - \lambda I)^{n+1}.
\]
(c) For $n \times n$ matrices $A, B, C, D$, if $A$ is similar to $C$ and $B$ is similar to $D$, then $A + B$ must be similar to $C + D$.

Proof. False. For example, let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \sim C = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & \phantom{1} \\ 1 & \phantom{-1} \end{bmatrix} \sim D = \begin{bmatrix} -1 & 1 \end{bmatrix}.$$  

But

$$A + B = \begin{bmatrix} 0 & 1 \\ 0 & \phantom{1} \end{bmatrix} \not\sim \begin{bmatrix} 0 & 0 \end{bmatrix} = C + D. \quad \square$$

(d) For every linear endomorphism $T : V \to V$ on a vector space $V$, a subspace $W$ of $V$ is $T$-invariant if and only if $W$ is $T^2$-invariant.

Proof. False. For example, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (y, 0)$ and $W = \text{Span}\{(1, 1)\}$. Then $T^2 = 0$ and $T^2(W) \subset W$. So $W$ is $T^2$-invariant. But

$$T(W) = \text{Span}\{T(1, 1)\} = \text{Span}\{(1, 0)\} \not\subset W$$

and $W$ is not $T$-invariant. \quad \square

(e) For every linear endomorphism $T : V \to V$ on a finite-dimensional vector spaces $V$,

$$K(T^4 - I) = K(T - I) + K(T + I) + K(T^2 + I).$$

Proof. True. Since

$$T^4 - I = (T - I)(T + I)(T^2 + I)$$

and

$$\gcd(x - 1, x + 1) = \gcd(x + 1, x^2 + 1) = \gcd(x^2 + 1, x - 1) = 1,$$

we conclude

$$K(T^4 - I) = K(T - I) + K(T + I) + K(T^2 + I)$$

by the generalized spectral theorem. \quad \square

(f) Two square matrices $A$ and $B$ are similar if and only if they have the same characteristic and minimal polynomials.
Proof. False. For example, let

\[ A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \]

Then both \( A \) and \( B \) have the same characteristic polynomial \( x^4 \) and the same minimal polynomial \( x^2 \). But

\[ 1 = \text{rank}(A) \neq \text{rank}(B) = 2 \]

and hence \( A \) and \( B \) are not similar. \( \square \)

(2) Solve the following system of ordinary differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 + x_2 \\
\frac{dx_2}{dt} &= x_1 + 2x_2 - x_3 \\
\frac{dx_3}{dt} &= x_1 + 2x_2
\end{align*}
\]

with \( x_1(0) = 1 \), \( x_2(0) = 2 \), \( x_3(0) = 3 \)

Solution. The characteristic polynomial of

\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & -1 \end{bmatrix} \]

is

\[
\det(xI - A) = \det \begin{bmatrix} x - 1 & -1 & 0 \\ -1 & x - 2 & 1 \\ -1 & -2 & x \end{bmatrix}
= \det \begin{bmatrix} 0 & x^2 - 3x + 1 & x - 1 \\ -1 & x - 2 & 1 \\ 0 & -x & x - 1 \end{bmatrix}
= \det \begin{bmatrix} x^2 - 3x + 1 & x - 1 \\ -x & x - 1 \end{bmatrix}
= (x - 1)(x^2 - 2x + 1) = (x - 1)^3.
\]

Hence the characteristic polynomial of \( tA \) is

\[ \det(xI - tA) = (x - t)^3. \]
By Taylor Theorem,
\[ e^z = (z-t)^3 q(z) + \frac{1}{2!} (e^z)^t (z-t)^2 + \frac{1}{1!} (e^z)^t (z-t) + (e^z) \]
for some entire function \( q(z) \). Therefore,
\[ e^{tA} = (tA - tI)^3 q(tA) + \frac{e^t}{2} (tA - tI)^2 + e^t (tA - tI) + e^t I \]
and the solution is
\[ x = e^{tA} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \frac{t^2 e^t}{2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + te^t \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2te^t + e^t \\ 2e^t \\ 2te^t + 3e^t \end{bmatrix} .\]

(3) Let \( A \) be a square matrix satisfying
\[ \{ \dim \text{Nul}(A - 2I)^k : k = 0, 1, \ldots \} = \{ 0, 2, 3, 3, \ldots \} \]
\[ \{ \dim \text{Nul}(A + I)^k : k = 0, 1, \ldots \} = \{ 0, 4, 5, 6, 6, \ldots \} \]
\[ \dim \text{Nul}(A - \lambda I)^k = 0 \text{ for all } \lambda \neq -1, 2 \in \mathbb{C} \text{ and all } k \in \mathbb{N} . \]

(a) Find the minimal and characteristic polynomials of \( A \).

Solution. Since
\[ \dim \text{Nul}(A - 2I) < \dim \text{Nul}(A - 2I)^2 = 3 = \dim \text{Nul}(A - 2I)^3 \]
\[ \dim \text{Nul}(A + I)^2 < \dim \text{Nul}(A + I)^3 = 6 = \dim \text{Nul}(A + I)^4 , \]
the minimal and characteristic polynomials of \( A \) are
\[ (x - 2)^2 (x + 1)^3 \text{ and } (x - 2)^3 (x + 1)^6 , \]
respectively.
(b) Find the Jordan Canonical Form of $A$.

Solution. For $\lambda = 2$, let $a_{2,k} = \dim \text{Nul}(A - 2I)^k$. We have

\[
\begin{align*}
    a_{2,k} : & \quad 0 & 2 & 3 & 3 \\
    b_{2,k} : & \quad 2 & 1 & 0 \\
    c_{2,k} : & \quad 1 & 1 \\
    & \quad J_{2,1} & J_{2,2}
\end{align*}
\]

For $\lambda = -1$, let $a_{-1,k} = \dim \text{Nul}(A + I)^k$. We have

\[
\begin{align*}
    a_{-1,k} : & \quad 0 & 4 & 5 & 6 & 6 \\
    b_{-1,k} : & \quad 4 & 1 & 1 & 0 \\
    c_{-1,k} : & \quad 3 & 0 & 1 \\
    & \quad J_{-1,1} & J_{-1,2} & J_{-1,3}
\end{align*}
\]

Thus, the Jordan Canonical Form of $A$ is

\[
\begin{bmatrix}
2 & & & & \ \\
2 & 1 & & & \\
2 & 1 & & & \\
& & & & \\
-1 & & & & \\
-1 & & & & \\
-1 & 1 & & & \\
-1 & & & & \\
-1 & & & & \\
\end{bmatrix}
\].

\[\square\]

(c) Find the minimal and characteristic polynomials of $A - 2A^{-1}$. Justify your answer.
Solution. Since \( \det(xI - A) = (x - 2)^3(x + 1)^6 \),

\[
\det(xI - (A - 2A^{-1})) = (x - g(2))^3(x - g(-1))^6 = (x - 1)^9
\]

for \( g(x) = x - 2x^{-1} \). So the characteristic polynomial of \( A - 2A^{-1} \) is \( (x - 1)^9 \).

Since

\[
\dim \text{Nul}(A - 2A^{-1} - I)^k = \dim \text{Nul}(A^{-k}(A^2 - A - 2I)^k)
= \dim \text{Nul}(A^2 - A - 2I)^k
= \dim \text{Nul}((A - 2I)^k(A + I)^k)
= \dim \text{Nul}(A - 2I)^k + \dim \text{Nul}(A + I)^k
\]

we have

\[
\{ \dim \text{Nul}(A - 2A^{-1} - I)^k : k = 0, 1, \ldots \} = \{0, 6, 8, 9, 9, \ldots \}.
\]

Hence the minimal polynomial of \( A - 2A^{-1} \) is \( (x - 1)^3 \). \( \square \)

(d) Find the Jordan Canonical Form of \( A - 2A^{-1} \). Justify your answer.

Solution. Let \( a_{1,k} = \dim \text{Nul}(A - 2A^{-1} - I)^k \). By (c), we have

\[
\begin{align*}
a_{1,k} : & \quad 0 \quad 6 \quad 8 \quad 9 \quad 9 \\
b_{1,k} : & \quad 6 \quad 2 \quad 1 \quad 0 \\
c_{1,k} : & \quad 4 \quad 1 \quad 1
\end{align*}
\]

\[
J_{1,1} \quad J_{1,2} \quad J_{1,3}
\]
Thus, the Jordan Canonical Form of $A - 2A^{-1}$ is
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}.
\]
\[
\square
\]

(4) Let $M_{m \times n}(\mathbb{R})$ be the vector space of $m \times n$ real matrices and $T : M_{2 \times 3}(\mathbb{R}) \rightarrow M_{2 \times 3}(\mathbb{R})$ be the linear endomorphism given by

\[T(A) = DA\]
where $D = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(a) Find the minimal and characteristic polynomials of $T$. Is $T$ diagonalizable? Justify your answer.

\textit{Solution.} Let $W_k$ be the subspace of $M_{2 \times 3}(\mathbb{R})$ defined by

\[W_k = \left\{ \begin{bmatrix} b_{ij} \end{bmatrix}_{2 \times 3} : b_{ij} = 0 \text{ for } j \neq k \right\}\]

for $k = 1, 2, 3$.

For every $B = \begin{bmatrix} 0 & \ldots & 0 & v_k & 0 & \ldots & 0 \end{bmatrix} \in W_k$,

\[T \begin{bmatrix} 0 & \ldots & 0 & v_k & 0 & \ldots & 0 \end{bmatrix} = D \begin{bmatrix} 0 & \ldots & 0 & v_k & 0 & \ldots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \ldots & 0 & Dv_k & 0 & \ldots & 0 \end{bmatrix} \in W_k.
\]

So $W_k$ is $T$-invariant for $k = 1, 2, 3$.

Let $C = \{E_{ij} : 1 \leq i \leq 2, 1 \leq j \leq 3\}$ be the standard basis of $M_{2 \times 3}(\mathbb{R})$. Then

\[W_k = \text{Span } C_k = \text{Span } \{E_{1k}, E_{2k}\}\]

for $k = 1, 2, 3$. Therefore,

\[M_{2 \times 3}(\mathbb{R}) = W_1 \oplus W_2 \oplus W_3.
\]

Let $T_k : W_k \rightarrow W_k$ be the restriction of $T$ to $W_k$.

For every $B = \begin{bmatrix} 0 & \ldots & 0 & v_k & 0 & \ldots & 0 \end{bmatrix} \in W_k$,

\[[B]_{C_k} = v_k \text{ and } [T_k(B)]_{C_k} = Dv_k.\]
Therefore, 
\[ [T_k]_{C_k \to C_k} = D. \]
So the minimal and characteristic polynomials of \( T_k \) are the same as those of \( D \), which are both \( f(x) = (x + 2)(x - 4) \).
So the minimal polynomial of \( T \) is
\[ \text{lcm}(f(x), f(x), f(x)) = f(x) = (x + 2)(x - 4) \]
and the characteristic polynomial of \( T \) is
\[ (f(x))^3 = (x + 2)^3(x - 4)^3. \]
Since \( (x + 2)(x - 4) \) is the minimal polynomial of \( T \),
\[ (T + 2I)(T - 4I) = 0 \Rightarrow K(T + 2I) + K(T - 4I) = V \]
for \( V = M_{2 \times 3}(\mathbb{R}) \), by the generalized spectral theorem. Therefore, \( T \) is diagonalizable. \( \square \)

(b) Find \( T^{2015}(A) \) for
\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \]

Solution. It suffices to find \( a \) and \( b \in \mathbb{R} \) such that
\[ x^n = (x + 2)(x - 4)q(x) + a(x + 2) + b(x - 4) \]
for some \( q(x) \in \mathbb{R}[x] \).
Setting \( x = 4 \), we obtain
\[ a = \frac{4^n}{4 + 2} = \frac{4^n}{6}. \]
Setting \( x = -2 \), we obtain
\[ b = \frac{(-2)^n}{-2 - 4} = \frac{(-2)^n}{6}. \]
So
\[ T^n(A) = D^nA = (D + 2I)(D - 4I)q(D)A + a(D + 2I)A + b(D - 4I)A \]
\[ = a(D + 2I)A + b(D - 4I)A \]
\[ = \frac{4^n}{6} \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} A - \frac{(-2)^n}{6} \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} A, \]
\[ T^n \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2^{2n-1} - (-2)^{n-1} & 2^{2n-1} + (-2)^{n-1} \\ 2^{2n-1} + (-2)^{n-1} & 2^{2n-1} - (-2)^{n-1} \end{bmatrix}, \]
\[ T^{2015} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2^{4029} - 2^{2014} & 2^{4030} \\ 2^{4029} + 2^{2014} & 2^{4029} - 2^{2014} \end{bmatrix}. \]

(5) Let \( A \) be an \( n \times n \) matrix satisfying that \( A^3 = 0 \).

(a) Show that
\[
\text{rank}(A) \leq \frac{2n}{3}
\]

Proof. Let \( a_k = \text{dim Nul}(A^k) \). Since \( A^3 = 0 \), \( a_3 = n \).
Since \( \{a_k - a_{k-1}\} \) is non-increasing,
\[
a_1 - a_0 \geq a_2 - a_1 \geq a_3 - a_2
\]
\[
\Rightarrow 3(a_1 - a_0) \geq (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2)
\]
\[
\Rightarrow 3a_1 \geq a_3
\]
\[
\Rightarrow 3 \text{dim Nul}(A) \geq n \Rightarrow \text{dim Nul}(A) \geq \frac{n}{3}
\]
\[
\Rightarrow n - \text{rank}(A) \geq \frac{n}{3} \Rightarrow \text{rank}(A) \leq \frac{2n}{3}.
\]

(b) Find a \( 6 \times 6 \) matrix \( A \) satisfying that \( A^3 = 0 \) and \( \text{rank}(A) = 4 \). Justify your answer.

Solution. Let \( A \) be a \( 6 \times 6 \) matrix with the required properties. Then
\[
\{\text{dim Nul}(A^k) : k = 0, 1, 2, ... \} = \{0, 2, a_2, 6, 6, ... \}
\]
since
\[
\text{dim Nul}(A) = 6 - \text{rank}(A) = 2 \quad \text{and}
\]
\[
\text{dim Nul}(A^3) = 6.
\]
Since \( 2 - 0 \geq a_2 - 2 \geq 6 - a_2 \), we must have \( a_2 = 4 \). So the Jordan Canonical Form of \( A \) must be
\[
J = \begin{bmatrix} \ J_{0,3} & \ J_{0,3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\]
We simply take \( A = J \).
(6) A complex square matrix $A$ is **unipotent** if $(A - I)^m = 0$ for some positive integer $m$.

(a) Show that $A$ is unipotent if and only if 1 is the only eigenvalue of $A$.

**Proof.** Suppose that $A$ is unipotent. Let $\lambda$ be an eigenvalue of $A$. Then $Av = \lambda v$ for some $v \neq 0$. Since $(A - I)^m = 0$ for some $m$,

$$0 = (A - I)^m v = (\lambda - 1)^m v$$

$$\Rightarrow (\lambda - 1)^m = 0 \Rightarrow \lambda = 1.$$

Therefore, 1 is the only eigenvalue of $A$. Suppose that 1 is the only eigenvalue of $A \in M_{n \times n}(\mathbb{C})$. Then the characteristic polynomial of $A$ must be

$$\det(x I - A) = (x - 1)^n.$$

By Cayley-Hamilton, $(A - I)^n = 0$ and hence $A$ is unipotent. $\square$

(b) Show that $A$ and $A^{-1}$ are similar for all unipotent matrices $A$.

**Proof.** Since $A$ is unipotent, 1 is the only eigenvalue of $A$ and $\det(x I - A) = (x - 1)^n$. Therefore,

$$\det(x I - A^{-1}) = (x - 1)^{-1} \cdot (x - 1)^n = (x - 1)^n$$

and hence 1 is also the only eigenvalue of $A^{-1}$. To show that $A$ and $A^{-1}$ are similar, it suffices to show that

$$\dim \text{Nul}(A - I)^m = \dim \text{Nul}(A^{-1} - I)^m$$

for all $m \in \mathbb{N}$. Since $A^{-1}$ is invertible,

$$\dim \text{Nul}(A^{-1} - I)^m = \dim \text{Nul}(A^{-m} (I - A)^m)$$

$$= \dim \text{Nul}(A^{-m} (-1)^m (A - I)^m)$$

$$= \dim \text{Nul}(A - I)^m$$

for all $m \in \mathbb{N}$. Hence $A$ and $A^{-1}$ are similar. $\square$