Math 325 Assignment #9
Due Apr. 10, 2015

(1) Solve the following systems of ODEs:

(a) \[
\begin{cases}
\frac{dx_1}{dt} = x_1 + 2x_2 \\
\frac{dx_2}{dt} = 4x_1 + 3x_2
\end{cases}
\]

(b) \[
\begin{cases}
\frac{dx_1}{dt} = x_1 + x_2 \\
\frac{dx_2}{dt} = -x_1 + 3x_2
\end{cases} \quad \text{with} \quad \begin{cases} x_1(0) = 1 \\
x_2(0) = 2 \end{cases}
\]

(2) Let \( T : P_3 \to P_3 \) be the linear endomorphism defined by
\[ T(f(x)) = 2f(x) + 3f'(x) + f''(x), \]
where \( P_3 \) is the vector space of polynomials in \( x \) of degree at most 3. Find \( T^{2015}(x^3) \).

(3) Which of the following statements are true and which are false? Justify your answer.

(a) Two square matrices \( A \) and \( B \) are similar if and only if \( \dim \operatorname{Nul}(A - \lambda I) = \dim \operatorname{Nul}(B - \lambda I) \) for all \( \lambda \).
(b) For every square matrix \( A \), \( A \) and \( A^T \) are always similar.
(c) There does not exist a linear endomorphism \( T : V \to V \) such that
\[ \{\operatorname{rank}(T^k) : k = 0, 1, 2, \ldots\} = \{8, 4, 3, 1, 1, \ldots\}. \]
(d) For an invertible linear endomorphism \( T : V \to V \) on a finite-dimensional vector space \( V \) over \( \mathbb{C} \), \( T \) is diagonalizable if and only if \( T^2 \) is diagonalizable.

(4) Show that two square matrices \( A \) and \( B \) are similar if and only if
\[ \operatorname{rank}(f(A)) = \operatorname{rank}(f(B)) \]
for all polynomials \( f(x) \in \mathbb{C}[x] \).

(5) Let \( A \) be an \( n \times n \) matrix satisfying \( A^m = 0 \) for some positive integer \( m \). Show that
\[ \operatorname{rank}(A) \leq \frac{(m-1)n}{m}. \]
(6) Let $T : V \to V$ be a linear endomorphism on a vector space $V$ and let $v_1$ and $v_2$ be two linearly independent generalized eigenvectors in $K(T - \lambda I)^m$. Suppose that

$$K(T - \lambda I)^{m-1} \cap \text{Span}\{v_1, v_2\} = \{0\}.$$ 

Show that

$$\text{Span}\{v_1, T(v_1), ..., T^{m-1}(v_1)\}$$

$$\cap \text{Span}\{v_2, T(v_2), ..., T^{m-1}(v_2)\} = \{0\}.$$

(7) Let $\{a_n : n = 0, 1, 2, \ldots\}$ be the Fibonacci sequence given by

$$a_{n+2} = a_{n+1} + a_n \text{ for all } n \geq 0 \text{ and } a_0 = a_1 = 1.$$

(a) Show that

$$\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

for all $n \geq 0$.

(b) Find a formula for $a_n$ (express it as a function of $n$).