Math 325 Assignment #8
Due Mar. 27, 2015

(1) Let \( T : V \to V \) be a linear endomorphism on a finite-dimensional vector space. Suppose that
\[
\{ \dim K(T - I)^k : k = 0, 1, \ldots \} = \{0, 2, 2, \ldots \}
\]
\[
\{ \dim K(T + I)^k : k = 0, 1, \ldots \} = \{0, 4, 6, 7, 7, \ldots \}
\]
\[\dim K(T - \lambda I)^k = 0 \text{ for all } \lambda \neq \pm 1 \text{ and all } k.\]
Do the following:
(a) Find the minimal and characteristic polynomials of \( T \). Justify your answer.
(b) Find the minimal and characteristic polynomials of \( T^2 + T \). Justify your answer.

(2) Find all dissimilar Jordan matrices \( J \) with minimal and characteristic polynomials \( x^3(x^2 - 1)^3 \) and \( x^4(x^2 - 1)^4(x + 1)^2 \), respectively.

(3) Which of the following statements are true and which are false? Justify your answer.
(a) If \( f(A) = 0 \) for a square matrix \( A \) and a polynomial \( f(x) \), then \( f(x) \) must be divisible by the characteristic polynomial of \( A \).
(b) Two square matrices with the same minimal and characteristic polynomials must be similar.
(c) If \( x^4 - x^2 \) is the minimal polynomial of a linear endomorphism \( T : V \to V \) on a finite dimensional vector space \( V \), then \( T \) is not diagonalizable.
(d) If \( x^2 + 5x + 4 \) is the minimal polynomial of a linear endomorphism \( T : V \to V \) on a finite dimensional vector space \( V \), then \( T \) is diagonalizable.

(4) Let \( T : M_{3\times3}(\mathbb{R}) \to M_{3\times3}(\mathbb{R}) \) be the linear endomorphism given by
\[ T(A) = 2A - 3A^T.\]
(a) Show that \( T \) is diagonalizable.
(b) Find \( T^{2015}(A) \) for
\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]
(5) Let $T : V \to V$ be a linear endomorphism satisfying

$$T^2 - 3T + 2I = 0.$$ 

(a) Find $a_n$ and $b_n \in \mathbb{R}$ such that

$$T^n = a_nT + b_nI$$ 

for all positive integers $n$.

(b) Find $c_n$ and $d_n \in \mathbb{R}$ such that

$$T^{-n} = c_nT + d_nI$$

for all positive integers $n$.

(6) Let $A$ be an $m \times m$ real matrix and $T : M_{m \times n}(\mathbb{R}) \to M_{m \times n}(\mathbb{R})$ be the linear endomorphism given by

$$T(B) = AB$$

for all $m \times n$ real matrices $B$. If the minimal and characteristic polynomials of $A$ are $f(x)$ and $g(x)$, respectively, find the minimal and characteristic polynomials of $T$. Justify your answer.