Solutions for Math 325 Assignment #7

(1) Let $T : V \rightarrow V$ be a linear endomorphism on a vector space $V$. Show that every one-dimensional $T$-invariant subspace of $V$ is spanned by an eigenvector of $T$.

Proof. Let $W$ be a one-dimensional $T$-invariant subspace of $V$ and let $W = \text{Span}\{v\}$. Then

$$T(W) \subset W \Rightarrow T(v) \in W = \text{Span}\{v\}.$$ 

Therefore, $T(v) = \lambda v$ for some scalar $\lambda$ and hence $v$ is an eigenvector of $T$. □

(2) Let $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ be a linear endomorphism given by

$$T(f(x)) = f(x) - f(2x - 1).$$

(a) Write $V = P_2(\mathbb{R})$ as a direct sum

$$V = W_1 \oplus W_2 \oplus W_3$$

of $T$-invariant subspaces of dimension at least one.

(b) Let $T_i : W_i \rightarrow W_i$ be the restriction of $T$ to $W_i$ for $i = 1, 2, 3$. Find the characteristic polynomial of each $T_i$.

Solution. Since $\dim W_i \geq 1$ and

$$\dim W_1 + \dim W_2 + \dim W_3 = \dim V = 3,$$

$\dim W_i = 1$ and $W_i$ are the eigenspaces of $T$.

Let $B = \{1, x, x^2\}$ be the standard basis of $P_2(\mathbb{R})$. Then

$$[T]_{B \leftarrow B} = [[T(1)]_B \ [T(x)]_B \ [T(x^2)]_B]$$

$$= [[0]_B \ [1 - x]_B \ [-1 + 4x - 3x^2]_B]$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & \ 4 \\ 0 & 0 & -3 \end{bmatrix}.$$ 

Therefore, $T$ has eigenvalues $0, -1, -3$ with eigenspaces

$W_1 = K(T) = \text{Span}\{1\}$

$W_2 = K(T + I) = \text{Span}\{1 - x\}$

$W_3 = K(T + 3I) = \text{Span}\{1 - 2x + x^2\}$.  


\[1\text{http://www.math.ualberta.ca/~xichen/math32515w/hw7sol.pdf}\]
Since
\[ T_1(f(x)) = 0 \text{ for } f(x) \in W_1, \]
\[ T_2(f(x)) = -f(x) \text{ for } f(x) \in W_2 \text{ and} \]
\[ T_3(f(x)) = -3f(x) \text{ for } f(x) \in W_3, \]
we conclude that
\[ \det(xI - [T_1]) = x \]
\[ \det(xI - [T_2]) = x + 1 \]
\[ \det(xI - [T_3]) = x + 3. \]

\(\square\)

(3) Which of the following statements are true and which are false? Justify your answer.

(a) For all linear endomorphisms \( T : V \to V \) and all subspaces \( W_1 \) and \( W_2 \) of \( V \), \( T(W_1 \cap W_2) = T(W_1) \cap T(W_2) \).

\textit{Proof.} False. For example, let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by \( T(x, y) = (x, 0) \), \( W_1 = \text{Span}\{(1, 0)\} \) and \( W_2 = \text{Span}\{(1, 1)\}. \) Then \( W_1 \cap W_2 = \{(0, 0)\}, \) and \( T(W_1 \cap W_2) = \{(0, 0)\}. \) But
\[ T(W_1) = \text{Span}\{T(1, 0)\} = \text{Span}\{(1, 0)\} \]
\[ T(W_2) = \text{Span}\{T(1, 1)\} = \text{Span}\{(1, 0)\} \]
and \( T(W_1) \cap T(W_2) = \text{Span}\{(1, 0)\} \neq T(W_1 \cap W_2). \) \(\square\)

(b) For all linear endomorphisms \( T : V \to V \) and all \( T \)-invariant subspaces \( W \) of \( V \), \( T^2(W) + T(W) \) is also \( T \)-invariant.

\textit{Proof.} True since \[ W \text{ \( T \)-inv } \Rightarrow T(W) \text{ \( T \)-inv} \]
\[ W \text{ \( T \)-inv } \Rightarrow T^2(W) \text{ \( T \)-inv} \] \[ \Rightarrow T^2(W) + T(W) \text{ \( T \)-inv}. \]
\(\square\)

(c) For all linear endomorphisms \( T_1 : V \to V \) and \( T_2 : V \to V \), \( R(T_1 + T_2) = R(T_1) + R(T_2). \)

\textit{Proof.} False. For example, let \( V = \mathbb{R}, T_1(x) = x \) and \( T_2(x) = -x. \) Then \( T_1 + T_2 = 0 \) and \( R(T_1 + T_2) = \{0\}. \) But \( R(T_1) = \mathbb{R}, R(T_2) = \mathbb{R} \) and \( R(T_1) + R(T_2) = \mathbb{R}. \) \(\square\)
(d) For every linear endomorphism $T : V \to V$, a subspace $W$ of $V$ is $T$-invariant if and only if it is $T^2$-invariant.

Proof. False. For example, let $V = \mathbb{R}^2$, $T(x, y) = (y, 0)$ and $W = \text{Span}\{(0, 1)\}$. Since $T^2 = 0$, every subspace of $V$ is $T^2$-invariant. So $W$ is $T^2$-invariant. On the other hand, $T(W) = \text{Span}\{(1, 0)\} \not\subset W$ and $W$ is not $T$-invariant. □

(4) For each of the following linear endomorphisms $T : V \to V$, find all its $T$-invariant subspaces:

(a) $V = \mathbb{R}^2$ and $T(x, y) = (x + y, x - y)$
(b) $V = P_2(\mathbb{R})$ and $T(g(x)) = g'(x) + 2g(x)$
(c) $V = M_{2 \times 2}(\mathbb{R})$ and $T(A) = -2AT$

where $P_n(\mathbb{R})$ is the vector space of polynomials in $\mathbb{R}[x]$ of degree $\leq n$ and $M_{n \times n}(\mathbb{R})$ is the vector space of $n \times n$ real matrices.

Solution. (a) Since the characteristic polynomial of $T$ is $x^2 - 2$, $T$ is diagonalizable and

$$V = K(T - \sqrt{2}I) \oplus K(T + \sqrt{2}I).$$

Every $T$-invariant subspace $W$ is

$$W = W_1 \oplus W_2$$

where $W_1$ is a $T$-invariant subspace of $K(T - \sqrt{2}I)$ and $W_2$ is a $T$-invariant subspace of $K(T + \sqrt{2}I)$. Therefore,

$$W = \begin{cases} 
\{0\} & \text{when } W_1 = W_2 = \{0\}, \\
K(T - \sqrt{2}I) & \text{when } W_1 = K(T - \sqrt{2}I) \text{ and } W_2 = \{0\}, \\
K(T + \sqrt{2}I) & \text{when } W_1 = \{0\} \text{ and } W_2 = K(T + \sqrt{2}I), \\
V & \text{when } W_1 = K(T - \sqrt{2}I) \text{ and } W_2 = K(T + \sqrt{2}I). 
\end{cases}$$

(b) Let $W$ be a $T$-invariant subspace of $V$. Since $W$ is $T$-invariant, it is $(T - 2I)$-invariant. For every $g(x) \in W$ of degree $\deg g(x) = m$,

$$W \supset \text{Span}\{g(x), (T - 2I)g(x), (T - 2I)^2g(x), ...\} = \text{Span}\{g(x), g'(x), g''(x), ..., g^{(m)}(x)\} = K((T - 2I)^{m+1}).$$

And since

$$\{0\} \subsetneq K(T - 2I) \subsetneq K((T - 2I)^2) \subsetneq K((T - 2I)^3) = V,$$

$W$ is one of $K((T - 2I)^m)$ for $m = 0, 1, 2, 3.$
(c) Note that $T$ is diagonalizable. It has two eigenvalues $\pm 2$ with eigenspaces

$$K(T - 2I) = \{ A \in M_{2\times2}(\mathbb{R}) : A^T = -A \}$$

$$K(T + 2I) = \{ A \in M_{2\times2}(\mathbb{R}) : A^T = A \}$$

It is diagonalizable since

$$\dim K(T - 2I) + \dim K(T + 2I) = \dim V.$$ 

So

$$V = K(T - 2I) \oplus K(T + 2I).$$

Every $T$-invariant subspace $W$ is

$$W = W_1 \oplus W_2$$

where $W_1$ is a $T$-invariant subspace of $K(T - 2I)$ and $W_2$ is a $T$-invariant subspace of $K(T + 2I)$. Since every subspace of $K(T - 2I)$ and $K(T + 2I)$ is $T$-invariant, a $T$-invariant subspace $W$ of $V$ is $W_1 \oplus W_2$ with $W_1 \subset K(T - 2I)$ and $W_2 \subset K(T + 2I)$. There are infinitely many $T$-invariant subspaces of $V$. \hfill \Box

(5) Let $T : V \to V$ be a linear endomorphism on a vector space $V$. Show that all subspaces $W$ of $V$ are $T$-invariant if and only if $T = cI$ for some constant $c$, where $I$ is the identity map.

**Proof.** If $T = cI$, then $T(W) = cW \subset W$ for all subspaces $W \subset V$ and hence every subspace $W$ of $V$ is $T$-invariant.

Suppose that every subspace of $V$ is $T$-invariant. Let $v_1$ and $v_2$ be two nonzero vectors in $V$. Since $\text{Span}\{v_1\}$ is $T$-invariant, $T(v_1) \in \text{Span}\{v_1\}$ and hence

$$T(v_1) = c_1v_1$$

for some constant $c_1$. Similarly,

$$T(v_2) = c_2v_2$$

$$T(v_1 + v_2) = c_3(v_1 + v_2)$$

for some constants $c_2$ and $c_3$. On the other hand,

$$T(v_1 + v_2) = T(v_1) + T(v_2) = c_1v_1 + c_2v_2$$

Therefore,

$$c_1v_1 + c_2v_2 = c_3(v_1 + v_2) \implies (c_1 - c_3)v_1 + (c_2 - c_3)v_2 = 0.$$
Suppose that $c_1 \neq c_2$. Then $c_1 - c_3$ and $c_2 - c_3$ are not all zero. So $v_1$ and $v_2$ are linearly dependent. Therefore, $v_1 = \lambda v_2$ for some constant $\lambda$. Therefore,

$$c_1 v_1 = T(v_1) = T(\lambda v_2) = \lambda T(v_2) = c_2(\lambda v_2) = c_2 v_1$$

and hence $c_1 = c_2$. Contradiction.

Therefore, $T(v_1) = cv_1$ and $T(v_2) = cv_2$ for all pairs of vectors $v_1$ and $v_2$. Consequently, $T(v) = cv$ for all $v \in V$. That is, $T = cI$. □

(6) Let $T : V \to V$ be a linear endomorphism on a vector space $V$. Let $v_1$ and $v_2$ be two vectors in $V$ satisfying

$$v_1 \in K(T - I), v_1 \neq 0$$

$$v_2 \in K((T + I)^2), v_2 \notin K(T + I).$$

Find the dimension of the smallest $T$-invariant subspace of $V$ containing both $v_1$ and $v_2$. Justify your answer.

**Solution.** Let $W$ be the smallest $T$-invariant subspace containing $v_1$ and $v_2$. Since $W$ is $T$-invariant, $W$ is $(T - I)$-invariant and $(T + I)$-invariant. So

$$W \supseteq W_1 = \text{Span}\{v_1, (T - I)v_1, \ldots\} = \text{Span}\{v_1\}$$

since $(T - I)^k v_1 = 0$ for $k \geq 1$ and

$$W \supseteq W_2 = \text{Span}\{v_2, (T + I)v_2, (T + I)^2v_2, \ldots\}$$

$$= \text{Span}\{v_2, (T + I)v_2\}$$

since $(T + I)^k v_2 = 0$ for $k \geq 2$. Therefore, $W \supseteq W_1 + W_2$.

Since $W_1$ is $(T - I)$-invariant, $W_1$ is $T$-invariant. Since $W_2$ is $(T + I)$-invariant, $W_2$ is $T$-invariant. So $W_1 + W_2$ is $T$-invariant. Consequently, $W_1 + W_2$ is the smallest subspace containing $v_1$ and $v_2$ and hence $W = W_1 + W_2$.

Since $W_1 \subset K(T - I)$, $W_2 \subset K((T + I)^2)$ and

$K(T - I) \cap K((T + I)^2) = \{0\}$,

$W_1 \cap W_2 = \{0\}$. So

$$\dim W = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

$$= \dim W_1 + \dim W_2.$$

Since $v_1 \neq 0$, $\dim W_1 = 1$.

Suppose that

$$c_1 v_2 + c_2 (T + I)v_2 = 0$$
for some scalars $c_1$ and $c_2$. Then
\[ c_1(T + I)v_2 + c_2(T + I)^2v_2 = 0 \Rightarrow c_1(T + I)v_2 = 0 \]
since $v_2 \in K((T+I)^2)$. And since $v_2 \not\in K(T+I)$, $(T+I)v_2 \neq 0$.
So $c_1 = 0$ and
\[ c_2(T + I)v_2 = 0. \]
Again, since $(T + I)v_2 \neq 0$, $c_2 = 0$. Therefore, $v_2$ and $(T + I)v_2$
are linearly independent and $\dim W_2 = 2$.
Consequently, $\dim W = 3$. \qed