Math 325 Assignment #5
Due Feb. 13, 2015

(1) Find \( \dim \text{Nul}(A - \lambda I)^k \) of the following square matrices \( A \) for all \( \lambda \in \mathbb{R} \) and \( k \geq 0 \):

a) \[
\begin{bmatrix}
4 & -1 \\
9 & -2
\end{bmatrix}
\]

b) \[
\begin{bmatrix}
3 & 1 & 0 \\
-1 & 1 & 0 \\
-1 & 0 & 2
\end{bmatrix}
\]

(2) Let \( A \) be a matrix similar to a Jordan matrix \( J \). Suppose that

\[
\{ \dim \text{Nul}(A - I)^k : k = 0, 1, \ldots \} = \{0, 1, 2, 3, 3, \ldots \}
\]

\[
\{ \dim \text{Nul}(A + I)^k : k = 0, 1, \ldots \} = \{0, 3, 5, 6, 6, \ldots \}
\]

\[
\dim \text{Nul}(A - \lambda I)^k = 0 \text{ for all } \lambda \neq \pm 1 \text{ and } k.
\]

Find \( J \).

(3) Which of the following statements are true and which are false? Justify your answer.

(a) For a square matrix \( A \), if \( v \) is an eigenvector of \( A \), then it is also an eigenvector of \( A^2 \).

(b) For a square matrix \( A \), if \( v \) is a generalized eigenvector of \( A \), then it is also a generalized eigenvector of \( A^2 \).

(c) For a square matrix \( A \), if \( \text{rank}(A^{2014}) = \text{rank}(A^{2016}) \), then \( \text{rank}(A^{2015}) = \text{rank}(A^{5102}) \).

(d) For all linear endomorphisms \( T : V \to V \), the sequence

\[
K(T) \subset K(T^2) \subset \ldots \subset K(T^N) \subset \ldots
\]

always stabilizes, i.e., there exists \( m \) such that

\[
K(T^m) = K(T^{m+1}) = \ldots = K(T^N) = \ldots
\]

(4) Show that

\[
\text{Nul}(A - \lambda_1 I)^m \cap \text{Nul}(A - \lambda_2 I)^n = \{0\}
\]

for all square matrices \( A \), \( \lambda_1 \neq \lambda_2 \) and positive integers \( m \) and \( n \).

(5) Let \( T : V \to V \) be a linear endomorphism on a vector space \( V \) of dimension \( n \). Suppose that \( T^N = 0 \) for some positive integer \( N \). Show that \( T^n = 0 \).

(6) For all square matrices \( A \), show that

\[
\text{rank}(A) + \text{rank}(A^3) \geq 2 \text{rank}(A^2).
\]