(1) Let $P_3$ be the vector space of real polynomials in $x$ of degree at most 3 and let $T: P_3 \rightarrow P_3$ be the linear transformation $T(f(x)) = xf'(x + 1)$.

(a) Find the matrix $[T]_{B \leftarrow B}$ representing $T$ under the basis $B = \{1, x, x^2, x^3\}$.

(b) Find the matrix $[T]_{B \leftarrow C}$ representing $T$ under the bases $B$ and $C = \{1, x + 1, x^2 + x + 1, x^3 + x^2 + x + 1\}$.

(c) Find the characteristic polynomial, eigenvalues and eigenvectors of $T$.

(d) Is $T$ diagonalizable? If it is, find a basis $D$ such that $[T]_{D \leftarrow D}$ is diagonal.

(2) Let $A$ be a square matrix with characteristic polynomial $x^3 - 4x$. Find the characteristic polynomial of $A^2 + 2A$. Justify your answer.

(3) Which of the following statements are true and which are false? Justify your answer.

(a) Two square matrices with the same characteristic polynomial must be similar.

(b) Two linear transformations $T_1: V \rightarrow W$ and $T_2: V \rightarrow W$ with the same kernel and range must be identical.

(c) A diagonalizable $n \times n$ matrix must have $n$ distinct eigenvalues.

(d) An $n \times n$ matrix cannot have more than $n$ eigenvectors.

(4) Let $P_2$ be the vector space of real polynomials in $t$ of degree at most 2 and let $T_1: \mathbb{R}^3 \rightarrow P_2$ and $T_2: P_2 \rightarrow \mathbb{R}^3$ be the linear transformations given by

$T_1(x_1, x_2, x_3) = x_1 + x_2t + x_3t^2$ and $T_2(f(t)) = (f(0) - f(1), f(1) - f(2), f(2) - f(3))$.

(a) Find $T_1 \circ T_2$ and $T_2 \circ T_1$.

(b) Find the kernels and ranges of $T_1$ and $T_2$.

(5) Let $A$ and $B$ be two square matrices satisfying $A^2 = B^2$ and let $\lambda$ be an eigenvalue of $A$. Show that either $\lambda$ or $-\lambda$ is an eigenvalue of $B$. 

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(6) Show that $A$ and $A^{-1}$ have the same eigenvectors for all invertible matrices $A$.

(7) Let $A$ be an $n \times n$ matrix satisfying $A^2 = 0$. Show that
\begin{equation*}
\text{rank}(A) \leq \frac{n}{2}.
\end{equation*}