A list of topics covered by the final:

- Upper triangularization.
- Jordan Blocks, Jordan Matrices and Jordan Canonical Forms.
- Jordan Canonical Forms of $2 \times 2$ matrices.
- Generalized Eigenvectors and Eigenspaces.
- Stabilization of Generalized Eigenspaces.
- Dimensions of Generalized Eigenspaces and Jordan Canonical Forms.
- Minimal Polynomials.
- Generalized Spectral Theorem.
- Invariant subspaces.
- Cayley-Hamilton Theorem.
- Computation of $A^n$.
- System of 1st order homogeneous linear ODEs with constant coefficients.
- Existence of Jordan Canonical Forms.

Important Algorithms:

- Upper triangularize a square matrix.
- Given the eigenvalues/characteristic polynomial/minimal polynomial of a square matrix $A$, find those of $f(A)$ for a rational function $f(x)$.
- Find the Jordan Canonical Form of a $2 \times 2$ matrix.
- Given $\dim \text{Nul}(A - \lambda I)^m$, find the Jordan Canonical Form, characteristic polynomial and minimal polynomial of $A$ and $f(A)$ for a rational function $f(A)$.
- Given the characteristic and minimal polynomials $A$, find all the possible Jordan Canonical Forms of $A$.
- Compute $A^n$.
- Compute $e^A$ and solve the system $x' = Ax$ of ODEs.

Important Theorems:

- Every complex square matrix can be upper-triangularized.
- The sequences $\{\text{Nul}(A - \lambda I)^m\}$ and $\{\text{Col}(A - \lambda I)^m\}$ stabilize.
- The minimal polynomial of a square matrix $A$ exists and is unique up to a scalar.
• Generalized Spectral Theorem: Given polynomials $f_1(x), f_2(x), \ldots, f_k(x)$ coprime to each other,

$$\text{Nul}(f_1(A)f_2(A) \ldots f_k(A)) = \text{Nul}(f_1(A)) \oplus \text{Nul}(f_2(A)) \oplus \ldots \oplus \text{Nul}(f_k(A)).$$

• If $W$ is a $T$-invariant subspace of a linear endomorphism $T : V \to V$ on a finite-dimensional vector space $V$, the characteristic and minimal polynomials of $T|_W$ divide those of $T$.

• For a linear endomorphism $T : V \to V$ on a finite-dimensional vector space $V$, if $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$ for $T$-invariant subspaces $W_i$, then the minimal polynomial of $T$ is the least common multiple of the minimal polynomials of $T_i = T|_{W_i}$ and the characteristic polynomial of $T$ is the product of the characteristic polynomials of $T_i$.

• Cayley-Hamilton Theorem: For every square matrix $A$,

$$f(A) = 0$$

for $f(x) = \det(xI - A)$.

• For every square matrix $A$, the dimensions of generalized eigenspaces $\text{Nul}(A - \lambda I)^m$ determine the characteristic and minimal polynomials of $A$: if

$$\dim \text{Nul}(A - \lambda_i I)^{m_i - 1} < \dim \text{Nul}(A - \lambda_i I)^{m_i} = n_i = \dim \text{Nul}(A - \lambda_i I)^{m_i + 1}$$

for eigenvalues $\lambda_i$ of $A$, then the characteristic and minimal polynomials of $A$ are

$$\prod_i (x - \lambda_i)^{n_i} \text{ and } \prod_i (x - \lambda_i)^{m_i},$$

respectively.

• For every square matrix $A$ over $\mathbb{C}$, there exists a Jordan matrix $J$ such that $A$ and $J$ are similar. The number of blocks $J_{\lambda,m}$ in $J$ is given by

$$2 \dim \text{Nul}(A - \lambda I)^m - \dim \text{Nul}(A - \lambda I)^{m-1} - \dim \text{Nul}(A - \lambda I)^{m+1}.$$ 

• Two matrices $A$ and $B$ are similar if and only if

$$\dim \text{Nul}(A - \lambda I)^m = \dim \text{Nul}(B - \lambda I)^m$$

for all $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$.

• For every square matrix $A$ and $\lambda \in \mathbb{C}$, $\{a_k = \dim \text{Nul}(A - \lambda I)^k\}$ is a non-decreasing and concave upward sequence.
Sample Problems:

(1) Which of the following statements are true and which are false? Justify your answer.

(a) If \( f(A) = 0 \) for a square matrix \( A \) and a polynomial \( f(x) \), then \( f(x) \) must be divisible by the characteristic polynomial of \( A \).

(b) Two square matrices with the same minimal and characteristic polynomials must be similar.

(c) For all linear endomorphisms \( T : V \to V \) and all \( T \)-invariant subspaces \( W \) of \( V \), \( T^2(W) + T(W) \) is also \( T \)-invariant.

(d) For every linear endomorphism \( T : V \to V \), a subspace \( W \) of \( V \) is \( T \)-invariant if and only if it is \( T^2 \)-invariant.

(e) The characteristic polynomial of an \( n \times n \) matrix \( A \) is a minimal polynomial of \( A \) if and only if \( A \) has \( n \) distinct eigenvalues.

(f) There does not exist a linear endomorphism \( T : V \to V \) such that
\[
\{ \text{rank}(T^k) : k = 0, 1, 2, \ldots \} = \{8, 4, 3, 1, 1, \ldots \}.
\]

(2) Let \( A \) be a square matrix with minimal and characteristic polynomials \( x^3(x^2 - 1)^3 \) and \( x^4(x^2 - 1)^4(x + 1)^2 \), respectively.

(a) Show that \( A \) is NOT diagonalizable.

(b) Find the characteristic and minimal polynomials of the matrix \((2I + A)^{-1}\). Justify your answer.

(c) Find all possible dissimilar Jordan canonical forms of \( A \).

(3) Let \( A \) be a square matrix satisfying
\[
\{ \dim \text{Nul}(A - I)^k : k = 0, 1, \ldots \} = \{0, 2, 2, \ldots \}
\]
\[
\{ \dim \text{Nul}(A + I)^k : k = 0, 1, \ldots \} = \{0, 4, 6, 7, 7, \ldots \}
\]
\[
\dim \text{Nul}(A - \lambda I)^k = 0 \text{ for all } \lambda \neq \pm 1 \text{ and all } k.
\]

(a) Find the minimal and characteristic polynomials of \( A \). Justify your answer.

(b) Find the Jordan canonical form of \( A \).

(c) Find the minimal and characteristic polynomials of \( A^3 - A \). Justify your answer.
(4) Solve the following systems of ODEs:

(a) \[
\begin{align*}
\frac{dx_1}{dt} &= x_1 + 2x_2 \\
\frac{dx_2}{dt} &= 4x_1 + 3x_2
\end{align*}
\]

(b) \[
\begin{align*}
\frac{dx_1}{dt} &= x_1 + x_2 \\
\frac{dx_2}{dt} &= -x_1 + 3x_2
\end{align*}
\]

with \[
\begin{align*}
x_1(0) &= 1 \\
x_2(0) &= 2
\end{align*}
\]

(5) Let \( T : V \rightarrow V \) be a linear endomorphism on a vector space \( V \).
Show that all subspaces \( W \) of \( V \) are \( T \)-invariant if and only if \( T = cI \) for some constant \( c \), where \( I \) is the identity map.

(6) Show that
\[
K(T^2 + T) + K(T^3 + T) = K(T^4 + T^3 + T^2 + T)
\]

for every linear endomorphism \( T : V \rightarrow V \) on a finite-dimensional vector space \( V \).

(7) Let \( T : M_{3\times3} (\mathbb{R}) \rightarrow M_{3\times3} (\mathbb{R}) \) be the linear endomorphism given by
\[
T(A) = 2A - 3A^T.
\]

(a) Show that \( T \) is diagonalizable.
(b) Find \( T^{2015} (A) \) for
\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]

(8) Let \( A \) be an \( n \times n \) matrix satisfying \( \text{rank}(A^m) = n - m \) for \( m = 1, 2, \ldots, n \).

(a) Show that 0 is the only eigenvalue of \( A \).
(b) Find the characteristic and minimal polynomials \( A \). Justify your answer.
(c) Find the Jordan canonical form of \( A \). Justify your answer.

(9) Show that \( A \) and \( A^T \) are similar for every square matrix \( A \).