(1) Consider the following equation
\[ x^2 + y + \sin(xy) = 0. \]
(a) Prove that this equation has a unique continuous solution \( y = y(x) \) such that \( y(0) = 0 \) in a small neighborhood of \((0, 0)\).
(b) Discuss the monotonicity of the function \( y = y(x) \) near \( x = 0 \).
(c) Does this equation have a unique solution \( x = x(y) \) such that \( x(0) = 0 \) near \((0, 0)\)? Why?

**Proof.** Let \( F(x, y) = x^2 + y + \sin(xy) \). Since \( F_x(x, y) = 2x + y \cos(xy) \) and \( F_y(x, y) = 1 + x \cos(xy) \), \( F_y(0, 0) = 1 \neq 0 \). Hence the equation has a unique solution \( y = y(x) \) satisfying \( y(0) = 0 \) in \( B_r(0) \) for some \( r > 0 \) by Implicit Function Theorem (IFT).

By IFT, \( y(x) \in C^1(B_r(0), \mathbb{R}) \) and \( y'(0) = -F_x(0, 0)/F_y(0, 0) = 0 \). We claim that \( y \in C^2(B_r(0), \mathbb{R}) \) for some \( r > 0 \). Note that
\[ F_x(x, y) + F_y(x, y)y'(x) = 0. \]
Let \( z(x) = y'(x) \). Then
\[
\begin{aligned}
F(x, y) &= 0 \\
F_x(x, y) + F_y(x, y)z(x) &= 0
\end{aligned}
\]
Let \( \phi : \mathbb{R}^3 \to \mathbb{R}^2 \) be the function
\[ \phi(x, y, z) = (F(x, y), F_x(x, y) + F_y(x, y)z). \]
Then the Jacobian of \( \phi \) is
\[
J_\phi = \begin{bmatrix}
F_x & F_y & 0 \\
F_{xx} + F_{xy}z & F_{xy} + F_{yy}z & F_y
\end{bmatrix}
\]
and hence \( \text{rank} J_\phi(0, 0, 0) = 2 \) with
\[
\det \begin{bmatrix}
F_y(0, 0) & 0 \\
F_{xy}(0, 0) & F_y(0, 0)
\end{bmatrix} \neq 0.
\]
Therefore, by IFT, the system of equations \( \phi(x, y, z) = 0 \) can be solved for \( y \) and \( z \). That is, it has solution \( y = y(x) \) and \( z = z(x) \in C^1(B_r(0), \mathbb{R}) \) with \( y(0) = z(0) = 0 \). This shows that
\( z(x) = y'(x) \in C^1(B_r(0), \mathbb{R}) \) and hence \( y(x) \in C^2(B_r(0), \mathbb{R}) \). And \( y''(x) \) satisfies

\[
(F_{xx}(x, y) + F_{xy}(x, y)y'(x)) + (F_{xy}(x, y) + F_{yy}(x, y)y'(x))y'(x) + F_y(x, y)y''(x) = 0
\]

and hence \( y''(0) = -F_{xx}(0, 0)/F_y(0, 0) = -2 \). Therefore, \( y(x) \) is concave downward at 0. By the second derivative test, \( y(x) \) is increasing on \((-r, 0)\) and decreasing on \((0, r)\).

Note that \( y(x) < 0 \) for \( x \in (-r, 0) \cup (0, r) \). For each \( y_0 \) satisfying that \( 0 > y_0 > \max(y(r), y(-r)) \), there exists \( x_1 \in (-r, 0) \) and \( x_2 \in (0, r) \) such that \( y(x_1) = y(x_2) = y_0 \). Therefore, there does not exist a solution \( x = x(y) \) near \((0, 0)\).

(2) Let \( U = B_r(p) \) for some \( r > 0 \) and \( p \in \mathbb{R}^2 \) and let \( g_1(x, y) \) and \( g_2(x, y) \) be two functions in \( C^1(U, \mathbb{R}) \) satisfying

\[
\frac{\partial g_1}{\partial x} = \frac{\partial g_2}{\partial y}
\]

on \( U \). Show that there exists \( g \in C^2(U, \mathbb{R}) \) such that

\[
\frac{\partial g}{\partial y} = g_1 \quad \text{and} \quad \frac{\partial g}{\partial x} = g_2,
\]

where \( C^l(U, \mathbb{R}) \) is the set of functions on \( U \) with continuous \( l \)-th partial derivatives.

Proof. We let

\[
G_1(x, y) = \int_0^y g_1(x, t)dt \quad \text{and} \quad G_2(x, y) = \int_0^x g_2(t, y)dt.
\]

Then, by Fundamental Theorem of Calculus (FTC), \( \partial G_1/\partial y = g_1 \) and \( \partial G_2/\partial x = g_2 \). And since \( \partial g_1/\partial x = \partial g_2/\partial y \),

\[
\frac{\partial^2 G_1}{\partial x \partial y} = \frac{\partial^2 G_2}{\partial x \partial y}
\]

in \( U \). Let \( G = G_1 - G_2 \). Then

\[
\frac{\partial}{\partial x} \left( \frac{\partial G}{\partial y} \right) = \frac{\partial^2 G}{\partial x \partial y} = 0.
\]

Therefore,

\[
\frac{\partial G}{\partial y} = f(y)
\]
for some continuous function \( f(y) \). Applying FTC again, we see that
\[
G(x, y) = F_1(x) + F_2(y)
\]
for some \( C^1 \) functions \( F_1 \) and \( F_2 \). Let
\[
g(x, y) = G_1(x, y) - F_1(x) = G_2(x, y) + F_2(y).
\]
Then \( \partial g / \partial y = g_1 \) and \( \partial g / \partial x = g_2 \).

(3) Let \( U \) be an open neighborhood of \( P_0 = (x_0, y_0) \) in \( \mathbb{R}^2 \) and let \( f \in C^2(U, \mathbb{R}) \) be a function satisfying
\[
f_{xx}(P_0)f_{yy}(P_0) - (f_{xy}(P_0))^2 \neq 0.
\]
Prove that for some \( r > 0 \) and \( (x, y) \in B_r(P_0) \), the following transformation
\[
\begin{align*}
\begin{cases}
u = f_x(x, y), \\
v = f_y(x, y), \\
w = -z + x f_x(x, y) + y f_y(x, y)
\end{cases}
\end{align*}
\]
has a unique inverse
\[
\begin{align*}
\begin{cases}
x = g_u(u, v) \\
y = g_v(u, v) \\
z = -w + u g_u(u, v) + v g_v(u, v)
\end{cases}
\end{align*}
\]
for some \( g \in C^2(V, \mathbb{R}) \) satisfying
\[
g_{uu}(Q_0)g_{vv}(Q_0) - (g_{uv}(Q_0))^2 \neq 0
\]
where \( V \) is an open neighborhood of \( Q_0 = (f_x(P_0), f_y(P_0)) \).

**Proof.** Let \( \phi : U \to \mathbb{R}^2 \) be the map given by \( \phi(x, y) = (f_x(x, y), f_y(x, y)) \). Then
\[
J_{\phi} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}
\]
with determinant \( \det(J_{\phi}) = f_{xx}f_{yy} - (f_{xy})^2 \). Since \( \det(J_{\phi}(P_0)) \neq 0 \), \( \phi \) has a local inverse \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) by Inverse Function Theorem (IFT). Let \( \varphi(u, v) = (g_1(u, v), g_2(u, v)) \). Then \( \varphi(Q_0) = P_0 \) and \( \varphi \in C^1(V, \mathbb{R}^2) \). Moreover, \( J_{\phi}(\varphi(q))J_{\varphi}(q) = I \) for all \( q \in V \). And since \( J_{\phi} \) is symmetric, \( J_{\varphi}(q) \) is symmetric for all \( q \in V \). It follows that
\[
\frac{\partial g_1}{\partial v} = \frac{\partial g_2}{\partial u}.
\]
and hence by (2), there is \( g \in C^2(V, \mathbb{R}) \) such that \( g_u = g_1 \) and \( g_v = g_2 \). So \( \varphi(u, v) = (g_u(u, v), g_v(u, v)) \) and hence

\[
g_{uu}(Q_0)g_{vv}(Q_0) - (g_{uv}(Q_0))^2 = \det J_\varphi(Q_0) \neq 0.
\]

Finally, since \( w = -z + xf_x(x, y) + yf_y(x, y) \) and \( \phi \circ \varphi = id \),

\[
z = -w + g_u(u, v)f_x(g_u(u, v), g_v(u, v)) + g_v(u, v)f_y(g_u(u, v), g_v(u, v))
\]

\[
= -w + ug_u(u, v) + vg_v(u, v).
\]

(4) Find the extreme values of the function \( f(x, y) = e^{-xy} \) on the region described by the inequality \( x^2 + 4y^2 \leq 1 \). Use Lagrange multipliers to treat the boundary case.

**Solution.** Since \( e^z \) is increasing in \( z \), \( f(x, y) = e^{-xy} \) achieves extremes wherever \( g(x, y) = xy \) achieves extremes. Since \( S = \{x^2 + 4y^2 \leq 1\} \) is compact and \( g(x, y) \) is continuous, \( g(x, y) \) has a maximum and minimum in \( S \).

Solve \( \nabla g = 0 \) and we obtain \( x = y = 0 \). So \( (x, y) = (0, 0) \) is the only critical point of \( g(x, y) \). If \( g(x, y) \) achieves the maximum or minimum at a point on \( \partial S \), it must have a local maximum or minimum at this point under the constraint \( x^2 + 4y^2 = 1 \) which can be computed by Lagrange Multipliers. We solve

\[
\begin{align*}
\nabla g(x, y) &= \lambda \nabla (x^2 + 4y^2 - 1) \\
x^2 + 4y^2 &= 1
\end{align*}
\]

\[
\Rightarrow \begin{cases}
2\lambda x = y \\
8\lambda y = x \\
x^2 + 4y^2 = 1
\end{cases} \Rightarrow \begin{cases}
x = \pm \frac{\sqrt{2}}{2} \\
y = \pm \frac{\sqrt{2}}{4}
\end{cases}
\]

We compare the values of \( g(x, y) \) at \( (x, y) = (0, 0) \) and \( (\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{4}) \) and conclude that \( g_{\text{max}} = 1/4 \) and \( g_{\text{min}} = -1/4 \) in \( S \). Correspondingly, \( f_{\text{min}} = e^{-1/4} \) and \( f_{\text{max}} = e^{1/4} \).

(5) Find the extreme values of the function \( f(x, y, z) = x^4 + y^4 + z^4 \) subject to the constraint \( xyz = 1 \).

**Solution.** The extreme values of \( f(x, y, z) = x^4 + y^4 + z^4 \) under the constraint \( xyz = 1 \) are the same as those of \( f(x, y, z) = x + y + z \) under the constraints \( xyz = 1 \) and \( x, y, z > 0 \).
Obviously, when \( x, y, z > 0 \), \( f(x, y, z) > 0 \) and
\[
\lim_{x \to \infty} f(x, y, z) = \lim_{y \to \infty} f(x, y, z) = \lim_{z \to \infty} f(x, y, z) = \infty.
\]
Therefore, \( f(x, y, z) \) does not have a maximum and does have a minimum in the set \( S = \{xyz = 1, x, y, z > 0\} \). And this minimum is a local minimum of \( f(x, y, z) \) under the constraint \( xyz = 1 \) and hence it can be computed by Lagrange Multipliers.

We solve the equations
\[
\begin{cases}
\nabla f(x, y, z) = \lambda \nabla (xyz - 1) \\
xyz = 1
\end{cases}
\]
\[
\Rightarrow
\begin{cases}
\lambda z = 1 \\
\lambda x = 1 \\
\lambda y = 1 \\
xyz = 1
\end{cases}
\Rightarrow
\begin{cases}
x = 1 \\
y = 1 \\
z = 1
\end{cases}
\]
Therefore, \( f(x, y, z) = x + y + z \) achieves the minimum 3 in \( S \) at \( x = y = z = 1 \). Correspondingly, \( f(x, y, z) = x^4 + y^4 + z^4 \) achieves the minimum 3 under the constraints \( xyz = 1 \) at \( x = y = z = 1 \).
Solutions for Math 317 Assignment #8

(1) Let \( f(x, y) \) be a function in \( C^2(\mathbb{R}^2, \mathbb{R}) \) satisfying

\[
\begin{align*}
f_{xx}(x, y) &\geq 0, \quad f_{yy}(x, y) \geq 0 \quad \text{and} \quad f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 \geq 0
\end{align*}
\]

for all \((x, y) \in \mathbb{R}^2\) and let \( g(x, y) = ax + by \) be a linear function in \( x \) and \( y \). Consider the extreme values of \( g(x, y) \) in \( D = \{ f(x, y) \leq 0 \} \).

Show that if \( g(x, y) \) achieves a local maximum at some point \((x_0, y_0) \in D\), i.e., \( g(x_0, y_0) \geq g(x, y) \) for some \( r > 0 \) and all \((x, y) \in B_r(x_0, y_0) \cap D\), then \( g(x, y) \) achieves a global maximum at \((x_0, y_0) \) in \( D\), i.e., \( g(x_0, y_0) \geq g(x, y) \) for all \((x, y) \in D\).

**Proof.** By the hypothesis on \( f \), \( \text{Hess}(f) \) is nonnegatively definite. We have proved before that such \( f \) is convex. That is,

\[
(1 - t)f(p) + tf(q) \geq f((1 - t)p + tq)
\]

for \( p, q \in \mathbb{R}^2 \) and all \( t \in [0, 1] \). We claim that \( D \) is convex.

Let \( p, q \in D \). Then \( f(p) \leq 0 \) and \( f(q) \leq 0 \). Hence

\[
f((1 - t)p + tq) \leq (1 - t)f(p) + tf(q) \leq 0
\]

for all \( t \in [0, 1] \). Therefore, \((1 - t)p + tq \in D \) for all \( t \in [0, 1] \). So \( D \) is convex.

Let \( p = (x_0, y_0) \) where \( g(x, y) \) has a local maximum such that \( g(p) \geq g(x, y) \) for some \( r > 0 \) and all \((x, y) \in B_r(p) \cap D\). Suppose that \( g(p) \) is not a global maximum. Then there exists \( q \in D \) such that \( g(q) > g(p) \). By the convexity of \( D \), \((1 - t)p + tq \in D \) for all \( t \in [0, 1] \). And since \( g(x, y) \) is linear,

\[
g((1 - t)p + tq) = (1 - t)g(p) + tg(q) > g(p)
\]

for all \( t \in (0, 1) \). When \( t < r/||p - q|| \),

\[
||(1 - t)p + tq - p|| = t||p - q|| < r
\]

and hence \((1 - t)p + tq \in B_r(p) \). Contradiction. \(\square\)

(2) Given \( r > 0 \), find the extreme values of the function

\[
f(x, y, z) = \ln x + 2 \ln y + 3 \ln z
\]

in the set \( \{(x, y, z) \in \mathbb{R}^3 : \ x, \ y, \ z > 0, \ x^2 + y^2 + z^2 = 6r^2\} \).

Prove that for \( a, b, c > 0 \)

\[
ab^2c^3 \leq 108 \left( \frac{a + b + c}{6} \right)^6.
\]
Proof. Since \( x^2 + y^2 + z^2 = 6r^2 \), \( x, y, z \leq \sqrt{6}r \) in the set \( S = \{(x, y, z) \in \mathbb{R}^3 : x, y, z > 0, x^2 + y^2 + z^2 = 6r^2 \} \). Hence
\[
f(x, y, z) \leq \ln(\sqrt{6}r) + 2\ln(\sqrt{6}r) + 3\ln(\sqrt{6}r) = 6\ln(\sqrt{6}r)
\]
in \( S \). Therefore, \( f(x, y, z) \) is bounded from above in \( S \). And since
\[
\lim_{x \to 0^+} f(x, y, z) = \lim_{y \to 0^+} f(x, y, z) = \lim_{z \to 0^+} f(x, y, z) = -\infty
\]
f\((x, y, z)\) has a maximum in \( S \) which can be found using Lagrange Multipliers. We solve the equations
\[
\begin{cases}
\nabla f(x, y, z) = \lambda \nabla (x^2 + y^2 + z^2 - 6r^2) \\
x^2 + y^2 + z^2 = 6r^2
\end{cases}
\]
\[
\Rightarrow \begin{cases}
2\lambda x^2 = 1 \\
2\lambda y^2 = 2 \\
2\lambda z^2 = 3
\end{cases}
\Rightarrow \begin{cases}
x = r \\
y = \sqrt{2}r \\
z = \sqrt{3}r
\end{cases}
\]
Therefore, \( f(x, y, z) \) achieves the maximum \( 6\ln r + \ln 2 + (3\ln 3)/2 \) when \((x, y, z) = (r, \sqrt{2}r, \sqrt{3}r)\). That is,
\[
\ln x + 2\ln y + 3\ln z \leq 6\ln r + \ln 2 + (3\ln 3)/2
\]
in \( S \). Or equivalently,
\[
x^2 y^4 z^6 \leq 108 \left( \frac{x^2 + y^2 + z^2}{6} \right)^6.
\]
Let \( a = x^2 \), \( b = y^2 \) and \( c = z^2 \). We obtain
\[
ab^2 c^3 \leq 108 \left( \frac{a + b + c}{6} \right)^6.
\]
\( \square \)

(3) Suppose that \( f \in C^1(\mathbb{R}, \mathbb{R}) \) and \( f(0) = 0 \). Evaluate
\[
\lim_{t \to 0} \frac{1}{\pi^3} \iiint_{x^2 + y^2 + z^2 \leq t^2} f(\sqrt{x^2 + y^2 + z^2}) \, dxdydz.
\]
**Solution.** Using spherical coordinates
\[
(x, y, z) = (r \cos \theta \cos \phi, r \cos \theta \sin \phi, r \sin \theta),
\]
we obtain
\[
\int \int \int f(\sqrt{x^2 + y^2 + z^2}) = \int_0^t \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} f(r)r^2 \cos \theta d\theta d\phi dr = 2 \int_0^t \int_0^{2\pi} r^2 f(r) d\phi dr = 4\pi \int_0^t r^2 f(r) dr.
\]

By L’Hospital, we have
\[
\lim_{t \to 0} \frac{1}{\pi t^4} \int \int \int_{x^2+y^2+z^2 \leq t^2} f(\sqrt{x^2 + y^2 + z^2}) dxdydz = \lim_{t \to 0} \frac{4t^2 f(t)}{4t^3} = \lim_{t \to 0} \frac{f(t)}{t} = \lim_{t \to 0} \frac{f'(t)}{1} = f'(0)
\]

(4) Let \( D \) be a subset in \( \mathbb{R}^2 \) given by
\[
D = \left\{ 2 \leq \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \leq 4 \right\}.
\]

Evaluate the integral
\[
\int_D \frac{1}{xy}.
\]

Solution. We use polar coordinates \((x, y) = (r \cos \theta, r \sin \theta)\). Then
\[
D = \left\{ (r, \theta) : \frac{\cos \theta}{4} \leq r \leq \frac{\cos \theta}{2}, \right.
\]
\[
\left. \frac{\sin \theta}{4} \leq r \leq \frac{\sin \theta}{2}, r \geq 0, -\pi \leq \theta \leq \pi \right\}.
\]
Since \( r \geq 0 \), \( \cos \theta \geq 0 \) and \( \sin \theta \geq 0 \) and hence \( \theta \in [0, \pi/2] \) in \( D \). So

\[
D = \left\{ (r, \theta) \in [0, \infty) \times [0, \pi/2] : \right. \\
\left. \max\left(\frac{\cos \theta}{4}, \frac{\sin \theta}{4}\right) \leq r \leq \min\left(\frac{\cos \theta}{2}, \frac{\sin \theta}{2}\right) \right\}.
\]

\[
= \left\{ (r, \theta) \in [0, \infty) \times [0, \pi/2] : \\
\frac{\cos \theta}{4} \leq r \leq \frac{\sin \theta}{2} \quad \text{for} \quad \theta \leq \pi/4 \\
\frac{\sin \theta}{4} \leq r \leq \frac{\cos \theta}{2} \quad \text{for} \quad \theta \geq \pi/4 \right\}
\]

Therefore,

\[
\int_D xy = \int_{\phi^{-1}(D)} r^2 \cos \theta \sin \theta \left| \det J_\phi \right| \\
= \int_{\tan^{-1}(1/2)}^{\pi/4} \int_{(\tan \theta)/4}^{(\sin \theta)/2} \frac{1}{\cos \theta \sin \theta} dr d\theta \\
+ \int_{\pi/4}^{\tan^{-1}(2)} \int_{(\tan \theta)/4}^{(\sin \theta)/2} \frac{1}{\cos \theta \sin \theta} dr d\theta \\
= 2 \int_{\tan^{-1}(1/2)}^{\pi/4} \frac{\ln(2 \tan \theta)}{\sin(2\theta)} d\theta + 2 \int_{\tan^{-1}(2)}^{\pi/2} \frac{\ln(2 \cot \theta)}{\sin(2\theta)} d\theta \\
= 4 \int_{\tan^{-1}(1/2)}^{\pi/4} \frac{\ln(2 \tan \theta)}{\sin(2\theta)} d\theta
\]

Since \( \sin(2\theta) = 2 \tan \theta / (1 + \tan^2 \theta) \) and \( d \tan \theta = (1 + \tan^2 \theta) d\theta \),

\[
4 \int_{\tan^{-1}(1/2)}^{\pi/4} \frac{\ln(2 \tan \theta)}{\sin(2\theta)} d\theta = 4 \int_{\tan^{-1}(1/2)}^{\pi/4} \frac{\ln(2 \tan \theta)}{2 \tan \theta} d\tan \theta \\
= 2 \int_{1/2}^{1} \frac{\ln(2t)}{t} dt \\
= 2 \int_{1/2}^{1} \ln(2t) d\ln(2t) \\
= 2 \int_{0}^{\ln 2} sds = (\ln 2)^2
\]
(5) Find the volume of the solid bounded by the surface

\[(x^2 + y^2 + z^2)^2 = a^2(x^2 + y^2 - z^2),\]

where \(a > 0\).

**Solution.** The solid is given by

\[S = \{(x^2 + y^2 + z^2)^2 \leq a^2(x^2 + y^2 - z^2)\}.

For \((x, y)\) fixed, we have

\[z^4 + (2(x^2 + y^2) + a^2)z^2 + (x^2 + y^2)^2 - a^2(x^2 + y^2) \leq 0\]

and hence

\[z^2 \leq -\frac{(2x^2 + 2y^2 + a^2) + a\sqrt{8x^2 + 8y^2 + a^2}}{2} = g(x, y)\]

in \(S\). So \(g(x, y) \geq 0\) and hence \(x^2 + y^2 \leq a^2\) in \(S\). Therefore,

\[\mu(S) = \int_S 1\]

\[= \int \int_{x^2 + y^2 \leq a^2} \left(\int_{-\sqrt{g(x,y)}}^{\sqrt{g(x,y)}} dz\right) dxdy\]

\[= 2 \int \int_{x^2 + y^2 \leq a^2} \sqrt{g(x,y)} dxdy\]

\[= \sqrt{2} \int_0^a \int_0^{2\pi} r \sqrt{a\sqrt{8r^2 + a^2} - (2r^2 + a^2)} d\theta dr\]

\[= 2\sqrt{2} \pi \int_0^a r \sqrt{a\sqrt{8r^2 + a^2} - (2r^2 + a^2)} dr\]

\[= 2\sqrt{2} \pi a^3 \int_0^1 r \sqrt{8r^2 + 1 - (2r^2 + 1)} dr\]

\[= \frac{\sqrt{2} \pi a^3}{4} \int_0^1 \sqrt{8r^2 + 1} \sqrt{8r^2 + 1 - (2r^2 + 1)} d\sqrt{8r^2 + 1}\]

\[= \frac{\sqrt{2} \pi a^3}{4} \int_1^3 u \sqrt{u - (u^2 + 3)/4} = \frac{\sqrt{2} \pi a^3}{8} \int_1^3 u \sqrt{1 - (u - 2)^2} du\]

\[= \frac{\sqrt{2} \pi a^3}{8} \int_{-1}^1 (u + 2) \sqrt{1 - u^2} du = \frac{\sqrt{2} \pi a^3}{8} \int_{-\pi/2}^{\pi/2} (2 + \sin t) \cos^2 t dt\]

\[= \frac{\sqrt{2} \pi a^3}{8} \left(\frac{t + \sin(2t)}{2} - \frac{\cos^3 t}{3}\right)\bigg|_{-\pi/2}^{\pi/2} = \frac{\sqrt{2} \pi^2 a^3}{8}\]