Math 317 Assignment #5
Due Mar 7, 2011

(1) Find the volume of the set
\[ B = \{ x^2 + y^2 + z^2 \leq 4, -3 \leq x + 2y + 2z \leq 3 \} \subset \mathbb{R}^3. \]

(2) The functions \( \Gamma(x) \) and \( \mathcal{B}(x, y) \) are defined by
\[
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = \lim_{a \to 0^+} \lim_{b \to \infty} \int_a^b t^{x-1}e^{-t}dt
\]
for \( x > 0 \) and
\[
\mathcal{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \lim_{a \to 0^+} \lim_{b \to 1^-} \int_a^b t^{x-1}(1-t)^{y-1}dt
\]
for \( x, y > 0 \). Show that
\[
\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\]
for all \( x, y > 0 \).

(3) Find \( \Gamma(1/2) \) and \( \int_{-\infty}^{\infty} e^{-x^2}dx \).

(4) Let \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by \( g(s, t) = (e^s\cos t, e^s\sin t) \) for \( (s, t) \in \mathbb{R}^2 \).
   (a) Prove that \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) is locally injective, but not globally injective.
   (b) Show that the set
   \[
   \{(e^s\cos t, e^s\sin t) : s \in (0, 1), t \in (0, 2\pi)\}
   \]
   is open.

(5) Let \( U \) be a nonempty open set in \( \mathbb{R}^N \), let \( f \in C^1(U, \mathbb{R}^N) \) and assume that \( \det J_f(x) \neq 0 \) for all \( x \in U \). Given \( y \notin f(U) \), define \( g : U \to \mathbb{R} \) be given by \( g(x) = \|y - f(x)\|^2 \). Prove that \( \nabla g(x) \neq 0 \) for all \( x \in U \).
Math 317 Assignment #6
Due Mar 14, 2011

(1) Let $U \subset \mathbb{R}^n$ be an open set. We call a map $f : U \to \mathbb{R}^m$ an open map if it sends open sets to open sets, i.e., $f(V)$ is open in $\mathbb{R}^m$ for all $V \subset U$ open. Show that $\|f(x)\|$ does not have a local maximum for an open map $f : U \to \mathbb{R}^m$.

(2) We call a map $f : U \to \mathbb{R}^m$ proper if $f^{-1}(K)$ is compact for every compact set $K \subset \mathbb{R}^m$. Suppose that $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ is proper and satisfies $\det J_f(x) \neq 0$ for all $x \in \mathbb{R}^N$. Then the set $f^{-1}(y)$ is finite for every $y \in \mathbb{R}^N$.

(3) Suppose that $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ satisfies
\[ \|f(x) - f(y)\| \geq 2\|x - y\| \]
for all $x, y \in \mathbb{R}^N$.
   (a) Show that $f : \mathbb{R}^N \to \mathbb{R}^N$ is injective.
   (b) Prove that $\det J_f(x) \neq 0$ for all $x \in \mathbb{R}^N$.
   (c) Show that $f(\mathbb{R}^N) = \mathbb{R}^N$.

(4) Show $f(x, y, z) = (xy, yz, xz)$ has a continuously differentiable inverse at $f(1, 1, 1) = (1, 1, 1)$. Find the derivative of the inverse at $(1, 1, 1)$.

(5) Show that the equation $(x^2 + y^2 + 2z^2)^{1/2} = \cos z$ can be solved uniquely for $y$ in terms of $x$ and $z$ near $(0, 1, 0)$. If $y = f(x, z)$ is the solution, find $\partial f/\partial x$ near the given point and compute the value of $f_x(0, 0)$. Can the same equation be solved for $z$ in terms of $x$ and $y$ near the same point?