Math 317 Assignment #3
Due Feb 7, 2011

(1) Compute the following integrals:
(a) \[ \int_{0}^{\pi/3} \tan^2 x \, dx; \]
(b) \[ \int_{0}^{\pi} e^x \sin x \, dx; \]
(c) \[ \int_{0}^{1} \frac{dx}{1 + x^3}. \]

(2) Show that
\[ \int_{0}^{1} x^m (1 - x)^n \, dx = \frac{m! n!}{(m + n + 1)!} \]
for all nonnegative integers \( m \) and \( n \).

(3) Let \( f_n : [a, b] \to \mathbb{R} \) be a sequence of functions that converge uniformly to a function \( f(x) \) on \([a, b]\). If \( f_n \) is Riemann integrable on \([a, b]\) for all \( n \), then \( f \) is also Riemann integrable on \([a, b]\) and
\[ \lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx = \int_{a}^{b} f(x) \, dx. \]

(4) Let \( f : [a, b] \to \mathbb{R} \) be a bounded function on \([a, b]\).
(a) If \( f^2 \) is Riemann integrable on \([a, b]\), does it follow that \( f \) is Riemann integrable on \([a, b]\)?
(b) If \( f^3 \) is Riemann integrable on \([a, b]\), does it follow that \( f \) is Riemann integrable on \([a, b]\)?
Prove or disprove these statements.

(5) Let \( f : [-\pi, \pi] \to \mathbb{R} \) be a continuous function. Show that
\[ \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx = 0 \]
Math 317 Assignment #4
Due Feb 14, 2011

(1) Find the volume (i.e. content) of the set
\[ T = \{(x, y, z) : (x^2 + y^2 + z^2 - R^2 - r^2)^2 \leq 4R^2(r^2 - z^2) \}, \]
where \( R > r > 0 \).

(2) Let \( S_1 \subset \mathbb{R}^m \) and \( S_2 \subset \mathbb{R}^n \) be bounded sets that have contents. Show that \( \mu(S_1 \times S_2) = \mu(S_1)\mu(S_2) \).

(3) Let \( f : D \to \mathbb{R} \) and \( g : D \to \mathbb{R} \) be two Riemann integrable functions on a bounded set \( D \subset \mathbb{R}^n \). Then
\[
\left( \int_{D} fg \right)^2 \leq \left( \int_{D} f^2 \right) \left( \int_{D} g^2 \right).
\]

(4) Let \( f : [a, b] \to \mathbb{R} \) be a continuously differentiable function on \([a, b]\) satisfying \( f(a) = f(b) = 0 \) and
\[
\int_{a}^{b} (f(x))^2 dx = 1.
\]
Then
\[
\int_{a}^{b} xf(x)f'(x) dx = -\frac{1}{2}
\]
and
\[
\left( \int_{a}^{b} (f'(x))^2 dx \right) \left( \int_{a}^{b} x^2(f(x))^2 dx \right) \geq \frac{1}{4}.
\]

(5) Let \( f : [0, \infty) \to \mathbb{R} \) be a continuous function such that \( \lim_{x \to \infty} f(x) \) exists. Show that
\[
\lim_{R \to \infty} \frac{1}{R} \int_{0}^{R} f(x) dx
\]
exists and
\[
\lim_{R \to \infty} \frac{1}{R} \int_{0}^{R} f(x) dx = \lim_{x \to \infty} f(x).
\]