Solutions for Math 317 Assignment #1

(1) Let $S_1 \subset \mathbb{R}^m$ and $S_2 \subset \mathbb{R}^n$ be two bounded sets. If either $\mu(S_1) = 0$ or $\mu(S_2) = 0$, then $\mu(S_1 \times S_2) = 0$ in $\mathbb{R}^{m+n}$.

Proof. WLOG, we assume that $\mu(S_1) = 0$. So for all $r > 0$, there exists compact intervals $I_1, I_2, ..., I_l$ such that $S_1 \subset I_1 \cup I_2 \cup ... \cup I_l$ and

$$\sum_{k=1}^l \mu(I_k) < r.$$ 

Since $S_2$ is bounded, $S_2 \subset J$ for some compact interval $J$. So

$$S_1 \times S_2 \subset \bigcup_{k=1}^l I_k \times J$$

and hence

$$\mu(S_1 \times S_2) \leq \sum_{k=1}^l \mu(I_k \times J) = \mu(J) \sum_{k=1}^l \mu(I_k) < r \mu(J).$$

As $r \to 0$, we see that $\mu(S_1 \times S_2) = 0$. □

(2) Let $p_n$ be a sequence of points in $\mathbb{R}^m$ such that $\lim_{n \to \infty} p_n$ exists. Then $S = \{p_n : n \in \mathbb{Z}^+\}$ has content zero in $\mathbb{R}^m$.

Proof. Let $p = \lim_{n \to \infty} p_n$. Then for all $r > 0$, there exists $N$ such that $p_n \in B_r(p)$ for all $n > N$.

Let $p = (x_1, x_2, ..., x_m)$. Then

$$B_r(p) \subset I_r = [x_1 - r, x_1 + r] \times [x_2 - r, x_2 + r] \times ... \times [x_m - r, x_m + r].$$

Hence $p_n \in I_r$ for all $n > N$. It follows that

$$S \subset S_N \cup I_r$$

where $S_N = \{p_n : n \leq N\}$. Since $S_N$ is a finite set, $\mu(S_N) = 0$. Therefore, $\mu(S) \leq \mu(S_N) + \mu(I_r) = 2^m r^m$.

As $r \to 0$, we obtain that $\mu(S) = 0$. □

(3) Let $f : D \to \mathbb{R}^m$ be a continuous function on a compact set $D \subset \mathbb{R}^n$. Show that the graph $G_f = \{(x, f(x)) : x \in D\}$ of $f$ has content zero in $\mathbb{R}^{m+n}$.
Proof. Since $D$ is compact, 

$$D \subset I = [-R, R]^n$$

for some $R$.

Since $f$ is continuous on $D$ and $D$ is compact, $f$ is uniformly continuous on $D$. Therefore, for all $r > 0$, there exists $d > 0$ such that $||f(x_1) - f(x_2)|| < r$ for all $x_1, x_2 \in D$ satisfying $||x_1 - x_2|| < d$.

Let $N$ be a positive integer such that $R/N < d/n$. We let $P$ be the partition of $I$ given by

$$P = \left\{-R + \frac{kR}{N} : k = 1, 2, \ldots, 2N - 1\right\}^n.$$

Suppose that $P$ subdivides $I$ into $I = \bigcup I_v$. For each $I_v$, $|I_v| = R/N$ and hence $||x_1 - x_2|| \leq nR/N < d$ for all $x_1, x_2 \in I_v$.

For each $I_v \cap D \neq \emptyset$, we choose a point $x_v \in I_v \cap D$. Then $||x - x_v|| < d$ for all $x \in I_v$ and hence $||f(x) - f(x_v)|| < r$ for all $x \in I_v \cap D$. Therefore,

$$G_f \subset \bigcup_v I_v \times [y_{1v} - r, y_{1v} + r] \times [y_{2v} - r, y_{2v} + r] \times \cdots \times [y_{mv} - r, y_{mv} + r]$$

and thus

$$\mu(G_f) \leq (2r)^m \sum_v \mu(I_v) = (2r)^m \mu(I)$$

where $f(x_v) = (y_{1v}, y_{2v}, \ldots, y_{mv})$. As $r \to 0$, we obtain that $\mu(G_f) = 0$. \hfill $\square$

(4) Show that a bounded convex set $S \subset \mathbb{R}^2$ with $\text{int}(S) = \emptyset$ has content zero. Also give an example to show this is false if we drop the convexity.

Proof. If $S = \emptyset$ or $S$ consists of a single point, $\mu(S) = 0$ obviously. Otherwise, suppose that $S$ contains at least two distinct points $p_1$ and $p_2$. We claim that $S$ is contained in the line $L = \{p_1 + t(p_2 - p_1) : t \in \mathbb{R}\}$ passing through $p_1$ and $p_2$.

Let $p_3 \neq p_1, p_2$ be another point of $S$. Since $S$ is convex, the triangle

$$\Delta p_1p_2p_3 = \{t_1p_1 + t_2p_2 + t_3p_3 : t_1 + t_2 + t_3 = 1, t_1, t_2, t_3 \geq 0\}$$

$$= \{t_1(p_1 - p_3) + t_2(p_2 - p_3) + p_3 : t_1 + t_2 \leq 1, t_1, t_2 \geq 0\}$$

is contained in $S$. 


If \( p_3 \not\in L \), then \( p_1 - p_3 \) and \( p_2 - p_3 \) are linearly independent. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be the map given by
\[
f(t_1, t_2) = t_1(p_1 - p_3) + t_2(p_2 - p_3) + p_3.
\]
Obviously, \( f \) is continuous. And since \( p_1 - p_3 \) and \( p_2 - p_3 \) are linearly independent, it is a bijection. We have \( f(1, 0) = p_1 \), \( f(0, 1) = p_2 \) and \( f(0, 0) = p_3 \). We let \( q_1 = (1, 0), q_2 = (0, 1) \) and \( q_3 = (0, 0) \). Then
\[
f^{-1}(\Delta p_1p_2p_3) = \Delta q_1q_2q_3 = \{(t_1, t_2) : t_1, t_2 \geq 0, t_1 + t_2 \leq 1\}
\]
and \( \text{int}(\Delta p_1p_2p_3) \neq \emptyset \) if and only if \( \text{int}(\Delta q_1q_2q_3) \neq \emptyset \).

Let \( q = (1/3, 1/3) \). It is easy to check that \( B_r(q) \subset \Delta q_1q_2q_3 \) for \( r < 1/6 \). So \( \text{int}(\Delta q_1q_2q_3) \neq \emptyset \) and \( \text{int}(\Delta p_1p_2p_3) \neq \emptyset \). Hence \( \text{int}(S) \neq \emptyset \). Contradiction. So \( p_3 \in L \) and \( S \subset L \).

Since \( S \) is bounded, \( S \subset [-R, R]^2 \) for some \( R \).

If \( L = \{(x, y) : y = kx + b\} \) for some constants \( k \) and \( b \), then \( S \subset G = \{(x, kx + b) : -R \leq x \leq R\} \) and hence \( \mu(S) = 0 \) by (3).

If \( L = \{(x, y) : x = c\} \) for some constant \( c \), then \( S \subset [-r + c, r + c] \times [-R, R] \) for all \( r > 0 \). Hence \( \mu(S) \leq 2Rr \) and we see that \( \mu(S) = 0 \) by letting \( r \to 0 \).

This is obviously false for \( S \) not convex. We let \( S = (\mathbb{Q} \times \mathbb{Q}) \cap ([0, 1] \times [0, 1]) \). Clearly, \( \text{int}(S) = \emptyset \) but \( \mu(S) = \mu(S) = \mu([0, 1] \times [0, 1]) = 1 \). \( \square \)

(5) We call a bounded set \( S \subset \mathbb{R}^n \) has content if \( \mu(\partial S) = 0 \). Show that the union, the intersection and the product of two sets with contents have contents.

Proof. We claim that \( \partial(S_1 \cup S_2) \subset \partial S_1 \cup \partial S_2 \) for all sets \( S_1, S_2 \subset \mathbb{R}^n \).

Since \( \overline{S_1} \cup \overline{S_2} \subset \overline{S_1 \cup S_2} \) and \( \partial(S_1 \cup S_2) \subset \overline{S_1} \cup \overline{S_2}, \partial(S_1 \cup S_2) \subset \overline{S_1 \cup S_2} \). Suppose that there is a point \( p \in \partial(S_1 \cup S_2) \) such that \( p \not\in \partial S_1 \cup \partial S_2 \). Since \( p \in \overline{S_1} \cup \overline{S_2} \) and \( \overline{S_i} = \text{int}(S_i) \cup \partial S_i \) for \( i = 1, 2 \), we must have \( p \in \text{int}(S_1) \cup \text{int}(S_2) \subset \text{int}(S_1 \cup S_2) \). Contradiction.

If \( \mu(\partial S_1) = 0 \) and \( \mu(\partial S_2) = 0 \), \( \mu(\partial(S_1 \cup S_2)) \leq \mu(\partial S_1) + \mu(\partial S_2) = 0 \) and hence \( S_1 \cup S_2 \) has content.

We claim that \( \partial(S_1 \cap S_2) \subset \partial S_1 \cup \partial S_2 \) for all sets \( S_1, S_2 \subset \mathbb{R}^n \).
Again, we have $\partial(S_1 \cap S_2) \subset \overline{S}_1 \cap \overline{S}_2$. Suppose that there is a point $p \in \partial(S_1 \cap S_2)$ such that $p \notin \partial S_1 \cup \partial S_2$. Since $p \in \overline{S}_i$ and 

$\overline{S}_i = \text{int}(S_i) \cup \partial S_i$

for $i = 1, 2$, we must have $p \in \text{int}(S_1) \cap \text{int}(S_2) = \text{int}(S_1 \cap S_2)$. Contradiction.

If $\mu(\partial S_1) = 0$ and $\mu(\partial S_2) = 0$, $\mu(\partial(S_1 \cap S_2)) \leq \mu(\partial S_1) + \mu(\partial S_2) = 0$ and hence $S_1 \cap S_2$ has content.

Finally, we claim that 

$\partial(S_1 \times S_2) = (\partial S_1 \times \overline{S}_2) \cup (\overline{S}_1 \times \partial S_2)$

for all $S_1, S_2 \subset \mathbb{R}^n$.

Since 

$\overline{S}_1 \times \overline{S}_2 = \overline{S}_1 \times \overline{S}_2$ and $\text{int}(S_1 \times S_2) = \text{int}(S_1) \times \text{int}(S_2)$,

we have

$\partial(S_1 \times S_2) = \overline{S}_1 \times \overline{S}_2 \setminus \text{int}(S_1 \times S_2)$

$= (\overline{S}_1 \times \overline{S}_2) \setminus (\text{int}(S_1) \times \text{int}(S_2))$

$= ((\text{int}(S_1) \cup \partial S_1) \times (\text{int}(S_2) \cup \partial S_2)) \setminus (\text{int}(S_1) \times \text{int}(S_2))$

$= (\text{int}(S_1) \times \partial S_2) \cup (\partial S_1 \times \text{int}(S_2)) \cup (\partial S_1 \times \partial S_2)$

$= (\text{int}(S_1) \times \partial S_2) \cup (\partial S_1 \times \partial S_2)$

$\cup ((\partial S_1 \times \text{int}(S_2)) \cup (\partial S_1 \times \partial S_2))$

$= (\partial S_1 \times \overline{S}_2) \cup (\overline{S}_1 \times \partial S_2)$.

If $\mu(\partial S_1) = 0$ and $\mu(\partial S_2) = 0$, then 

$\mu(\partial(S_1 \times S_2)) \leq \mu(\partial S_1 \times \overline{S}_2) + \mu(\overline{S}_1 \times \partial S_2) = 0$

by (1) and hence $S_1 \times S_2$ has content. \qed
Solutions for Math 317 Assignment #2

(1) Find the following integrals by computing the limits of the corresponding Riemann sums:

(a) \( \int_{-1}^{1} x(1 - x)dx; \)

(b) \( \int_{D} xydxdy \) where \( D = [-1, 1] \times [0, 2] \).

Here you may use the identity \( \sum_{k=1}^{n} k^2 = n(n + 1)(2n + 1)/6. \)

**Solution.**

(a) Let \( n \in \mathbb{Z}^+ \) and \( P_n = \left\{ \frac{1 - k}{n} : k = 1, 2, ..., 2n - 1 \right\} \).

A corresponding Riemann sum is

\[
S(x(1 - x), P_n) = \frac{1}{n^2} \sum_{k=1}^{2n} \frac{k}{n} \left( 1 - \frac{k}{n} \right)
\]
\[
= \frac{1}{n^2} \sum_{k=1}^{2n} k - \frac{1}{n^3} \sum_{k=1}^{2n} k^2
\]
\[
= \frac{2n(2n + 1)}{2n^2} - \frac{2n(2n + 1)(4n + 1)}{6n^3}
\]

and hence

\[
\int_{-1}^{1} x(1 - x)dx = \lim_{n \to \infty} S(x(1 - x), P_n) = 2 - \frac{8}{3} = -\frac{2}{3}\]

(b) Let \( n \in \mathbb{Z}^+ \) and

\( P_n = \left\{ -1 + \frac{k}{n} : k = 1, 2, ..., 2n - 1 \right\} \times \left\{ \frac{l}{n} : l = 1, 2, ..., 2n - 1 \right\} \).

A corresponding Riemann sum is

\[
S(xy, P_n) = \frac{1}{n^2} \sum_{k=1}^{2n} \sum_{l=1}^{2n} \left( -1 + \frac{k}{n} \right) \frac{l}{n}
\]
\[
= \frac{1}{n^4} \sum_{k=1}^{2n} \sum_{l=1}^{2n} (k - n)l
\]
\[
= \frac{1}{n^4} \left( \sum_{k=1}^{2n} (k - n) \right) \left( \sum_{l=1}^{2n} l \right)
\]
\[
= \frac{2n + 1}{n^2}\]
and hence
\[ \int_D xy \, dx \, dy = \lim_{n \to \infty} S(xy, P_n) = 0. \]

(2) Show that every monotonic function \( f : [a, b] \to \mathbb{R} \) is Riemann integrable on \([a, b]\).

Proof. Let \( n \in \mathbb{Z}^+ \) and
\[ P_n = \left\{ a + \frac{k(b-a)}{n} : k = 1, 2, \ldots, n-1 \right\} \]
which subdivides \([a, b]\) into
\[ [a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \ldots \cup [x_{n-1}, x_n] \]
where \( x_k = a + k(b-a)/n \) for \( k = 0, 1, 2, \ldots, n \). WLOG, we assume that \( f(x) \) is nondecreasing. Then
\[ U(f, P_n) - L(f, P_n) = \frac{b-a}{n} \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = \frac{b-a}{n} (f(b) - f(a)) \]
and hence
\[ \lim_{n \to \infty} (U(f, P_n) - L(f, P_n)) = 0. \]
Therefore, \( f \) is Riemann integrable on \([a, b]\). \( \square \)

(3) If \( f : D \to \mathbb{R} \) is Riemann integrable on a bounded set \( D \subset \mathbb{R}^n \), then both
\[ f_+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases} \]
and
\[ f_-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases} \]
are Riemann integrable on \( D \).

Proof. We observe that
\[ f_+(x) = \frac{|f(x)| + f(x)}{2} \]
and
\[ f_-(x) = \frac{|f(x)| - f(x)}{2}. \]
Since \( f \) is Riemann integrable on \( D \), \(|f|\) is Riemann integrable on \( D \). Therefore, both \( f_+(x) \) and \( f_-(x) \) are Riemann integrable on \( D \). \( \square \)
(4) If \( f : D \to \mathbb{R} \) and \( g : D \to \mathbb{R} \) are Riemann integrable on a bounded set \( D \subset \mathbb{R}^n \), then \( fg \) is Riemann integrable on \( D \).

**Proof.** Let \( I \) be a compact interval containing \( D \). We extend \( f \) and \( g \) by zero to functions on \( I \).

Since \( f \) and \( g \) are Riemann integrable on \( D \), for every \( r > 0 \), there exist partitions \( P_r \) and \( Q_r \) of \( I \) such that
\[
U(f, P_r) - L(f, P_r) < r \quad \text{and} \quad U(g, Q_r) - L(g, Q_r) < r.
\]
We let \( T_r \supset P_r \cup Q_r \) be a refinement of both \( P_r \) and \( Q_r \). Then
\[
U(f, T_r) - L(f, T_r) \leq U(f, P_r) - L(f, P_r) < r
\]
and
\[
U(g, T_r) - L(g, T_r) \leq U(g, Q_r) - L(g, Q_r) < r.
\]

Also since \( f \) and \( g \) are Riemann integrable on \( D \), they are bounded and hence there are constants \( A \) and \( B \) such that
\[
|f(x)| \leq A \quad \text{and} \quad |g(x)| \leq B \quad \text{for all} \quad x \in I.
\]
Suppose \( T_r \) subdivides \( I \) into \( I = \bigcup I_v \). For \( x_1, x_2 \in I_v \),
\[
|f(x_1)g(x_1) - f(x_2)g(x_2)| = |f(x_1)(g(x_1) - g(x_2)) + (f(x_1) - f(x_2))g(x_2)| \\
\leq |f(x_1)||g(x_1) - g(x_2)| + |f(x_1) - f(x_2)||g(x_2)| \\
\leq A\sup_{x \in I_v} g(x) - \inf_{x \in I_v} g(x) + B\sup_{x \in I_v} f(x) - \inf_{x \in I_v} f(x)
\]
and it follows that
\[
U(fg, T_r) - L(fg, T_r) \\
\leq A(U(g, T_r) - L(g, T_r)) + B(U(f, T_r) - L(f, T_r)) \\
< (A + B)r
\]
and hence \( fg \) is Riemann integrable on \( D \). \( \square \)

(5) Let \( S_1 \) and \( S_2 \) be two disjoint sets in \( \mathbb{R}^n \) with contents. Then
\[
\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2).
\]

**Proof.** Let \( \chi_{S_1 \cup S_2}, \chi_{S_1} \) and \( \chi_{S_2} \) be the characteristic functions of \( S_1 \cup S_2, S_1 \) and \( S_2 \), respectively. Since \( S_1 \) and \( S_2 \) have contents, \( \chi_{S_1} \) and \( \chi_{S_2} \) are Riemann integrable on every compact interval \( I \supset S_1 \cup S_2 \) and
\[
\int_I \chi_{S_i} = \mu(S_i)
\]
for $i = 1, 2$. And since $\chi_{S_1 \cup S_2} = \chi_{S_1} + \chi_{S_2}$, $\chi_{S_1 \cup S_2}$ is Riemann integrable on $I$ and
\[
\int_I \chi_{S_1 \cup S_2} = \int_I \chi_{S_1} + \int_I \chi_{S_2} = \mu(S_1) + \mu(S_2).
\]
And since $\chi_{S_1 \cup S_2}$ is Riemann integrable on $I$, $S_1 \cup S_2$ has content and
\[
\int_I \chi_{S_1 \cup S_2} = \mu(S_1 \cup S_2).
\]
Therefore,
\[
\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2).
\]