Math 311 Assignment #8
Due Nov. 18, 2013

(1) Compute the integrals of the following functions along the curves
$C_1 = \{ |z| = 1 \}$ and $C_2 = \{ |z - 2| = 1 \}$, both oriented counter-clockwise:

(a) $\frac{1}{2z^2 - z^3}$;

(b) $\frac{\cosh z}{(2z - z^2)^2}$.

(2) Compute the integral
$$\int_0^\pi \frac{\sin^2 x}{2 - \cos x} \, dx$$

(3) Compute the integral
$$\int_0^\infty \frac{\cos(tx)}{x^4 + x^2 + 1} \, dx$$
for some $t \geq 0$.

(4) Compute the same integral in the previous problem for some $t \leq 0$.

(5) Compute the integral
$$\int_0^\infty \frac{x \, dx}{x^5 + 1}.$$

(6) Let $f(z)$ be an entire function. Show that $f(z)$ is a constant if
$|f(z)| \leq \sqrt{|z| + 1}$ for all $z \in \mathbb{C}$.

(7) Let $f(z) = u(x, y) + iv(x, y)$ be an entire function, where
$u(x, y) = \text{Re}(f(z))$ and $v(x, y) = \text{Im}(f(z))$.

Show that $f(z)$ is a constant if $u(x, y) + v(x, y) \geq 0$ for all $z = x + yi \in \mathbb{C}$.
(8) Let $a_1, a_2, ..., a_n$ be $n \geq 2$ distinct complex numbers. Compute the contour integral

$$\int_{|z|=R} \frac{dz}{(z-a_1)(z-a_2)\ldots(z-a_n)}$$

for some $R > \max(|a_1|, |a_2|, ..., |a_n|)$.

(9) Use the result of the previous problem to prove the identity

$$\sum_{j=1}^{n} \frac{1}{\prod_{1 \leq k \neq j \leq n} (a_j - a_k)} = 0$$

for all $n \geq 2$ distinct complex numbers $a_1, a_2, ..., a_n$. For example, when $n = 3$, the above identity is

$$\frac{1}{(a_1 - a_2)(a_1 - a_3)} + \frac{1}{(a_2 - a_3)(a_2 - a_1)} + \frac{1}{(a_3 - a_1)(a_3 - a_2)} = 0.$$

(10) Let $f(z)$ be an entire function satisfying

$$|f(2z)| \leq 2|f(z)|$$

for all complex numbers $z$. Show that either $f(z) \equiv az$ or $f(z) \equiv b$ for some constants $a, b \in \mathbb{C}$. 