(1) Compute the following contour integrals:
(a) \( \int_\gamma (z^3 + 2z)\,dz \), where \( \gamma(t) = (1 - t^2) + ti \) for \( 0 \leq t \leq 1 \).
(b) \( \int_\gamma e^{3z}\,dz \), where \( \gamma(t) = 2 + e^{it} \) for \( 0 \leq t \leq \pi/2 \).
(c) \( \int_\gamma z^2 \sin zdz \), where \( \gamma \) is the line segment from \( \pi \) to \( i \).

Solution. (a) Since \((z^4/4 + z^2)' = z^3 + 2z\),
\[
\int_\gamma (z^3 + 2z)\,dz = \left. \left( \frac{z^4}{4} + z^2 \right) \right|_1^i = -2.
\]
(b) Since \((e^{3z}/3)' = e^{3z}\),
\[
\int_\gamma e^{3z}\,dz = \left. \frac{e^{3z}}{3} \right|_2^i = \frac{e^6 \cos 3 - e^9}{3} + \frac{e^6 \sin 3}{3} i.
\]
(c) Using integration by parts
\[
\int_\gamma z^2 \sin zdz = - \int z^2 d(\cos z) = \int \cos zd(z^2) - z^2 \cos z
\]
\[
= 2 \int z \cos zdz - z^2 \cos z = 2 \int zd(sin z) - z^2 \cos z
\]
\[
= 2z \sin z - 2 \int \sin zdz - z^2 \cos z
\]
\[
= 2z \sin z + (2 - z^2) \cos z
\]
we obtain \((2z \sin z + (2 - z^2) \cos z)' = z^2 \sin z\). Therefore,
\[
\int_\gamma z^2 \sin zdz = (2z \sin z + (2 - z^2) \cos z)\bigg|_\pi^i
\]
\[
= \frac{e}{2} + \frac{5}{2e} + 2 - \pi^2
\]
\[
\square
\]

(2) Use the Euler formula to express the real integral
\[
\int_{-\pi}^{\pi} \frac{\sin \theta}{2 - \cos \theta} \,d\theta
\]
\[1\text{http://www.math.ualberta.ca/~xichen/math31113f/hw7sol.pdf}\]
as a contour integral
\[ \int_{|z|=1} f(z) dz. \]

**Solution.** Let \( z = e^{i\theta} \). Then \( dz = ie^{i\theta} d\theta \), \( d\theta = -iz^{-1}dz \) and
\[
\frac{\sin \theta}{2 - \cos \theta} = \frac{e^{i\theta} - e^{-i\theta}}{i(4 - (e^{i\theta} + e^{-i\theta}))} = \frac{z - z^{-1}}{z^2 - 1} \frac{z^2 - 1}{i(4z - z^2 - 1)}.
\]
Therefore,
\[
\int_{-\pi}^{\pi} \frac{\sin \theta}{2 - \cos \theta} d\theta = \int_{|z|=1} \frac{z^2 - 1}{z(z^2 - 4z + 1)} dz.
\]

(3) Let \( f(z) \) be a complex function defined in a connected open set \( G \subset \mathbb{C} \). Show that if both \( f(z) \) and \( \overline{f(z)} \) are holomorphic in \( G \), \( f(z) \) must be constant in \( G \).

**Proof.** We have proved that if \( f(z) \) is differentiable at \( z_0 \) and \( f'(z_0) \neq 0 \), \( \overline{f(z)} \) is not differentiable at \( z_0 \). Therefore, \( f'(z_0) = 0 \) for all \( z_0 \in G \). And since \( G \) is connected, \( f(z) \) must be constant in \( G \).

(4) Compute
\[ \int_{-1}^{1} z^{2i} dz \]
where the integrand denote the principal branch
\[ z^{2i} = \exp(2i \text{Log } z) \]
of \( z^{2i} \) and where the path of integration is any continuous curve from \( z = -1 \) to \( z = 1 \) that, except for its starting and ending points, lies below the real axis.
Solution. Note that $z^{2i+1}/(2i+1)$ is an anti-derivative of $z^{2i}$ outside the branch locus $(-\infty,0]$. So

$$\int_{-1}^{1} z^{2i} dz = \frac{z^{2i+1}}{2i+1} \bigg|_1^{z=1} - \lim_{z \to -1} \frac{z^{2i+1}}{2i+1} \bigg|_{\text{Im}(z)<0}$$

$$= \frac{1}{2i+1} - \exp((2i+1)(-\pi i)) = \frac{1 + e^{2\pi}}{2i+1} = \frac{1 + e^{2\pi}}{5} (1 - 2i)

$$

□

(5) Compute the same integral in the previous problem if the path of the integration is any continuous curve from $z = -1$ to $z = 1$ that, except the starting and ending points, lies above the real axis.

Solution. As in (4), we have

$$\int_{-1}^{1} z^{2i} dz = \frac{z^{2i+1}}{2i+1} \bigg|_1^{z=1} - \lim_{z \to -1} \frac{z^{2i+1}}{2i+1} \bigg|_{\text{Im}(z)>0}$$

$$= \frac{1}{2i+1} - \exp((2i+1)(\pi i)) = \frac{1 + e^{-2\pi}}{2i+1} = \frac{1 + e^{-2\pi}}{5} (1 - 2i)

$$

□

(6) Apply Cauchy Integral Theorem to show that

$$\int_{C} f(z) dz = 0$$

when $C$ is the unit circle $|z-2| = 1$, in either direction, and when

(a) $f(z) = \frac{1}{z^2+1}$;

(b) $f(z) = e^{\text{cot} z}$;

(c) $f(z) = \text{Log}(z+i)$.

Solution. By Cauchy Integral Theorem, $\int_{|z-2|=1} f(z) dz = 0$ if $f(z)$ is analytic on and inside the circle $|z-2| = 1$. Hence it is enough to show that $f(z)$ is analytic in $\{|z-2| \leq 1\}$. 
(a) $f(z)$ is analytic in $\{z \neq \pm i\}$, $|\pm i - 2| > 1$ and hence $f(z)$ is analytic in $\{|z - 2| \leq 1\}$.

(b) $f(z)$ is analytic in $\{z : \sin z \neq 0\} = \{z \neq n\pi, n \in \mathbb{Z}\}$. Since $|n\pi - 2| > 1$ for all integers $n$, $f(z)$ is analytic in $\{|z - 2| \leq 1\}$.

(c) Log($z$) is analytic in $\mathbb{C}\backslash(-\infty, 0]$ and hence Log($z + i$) is analytic in $\mathbb{C}\{z : z = x - i, x \in (-\infty, 0]\}$. Since $|x - i - 2| > 1$ for all $x \leq 0$, $f(z)$ is analytic in $\{|z - 2| \leq 1\}$.

(7) Let $C_1$ denote the positively oriented boundary of the curve given by $|x| + |y| = 4$ and $C_2$ be the positively oriented circle $|z| = 2$. Apply Cauchy Integral Theorem to show that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

when

(a) $f(z) = \frac{z + 1}{z^2 + z + 1}$.

(b) $f(z) = \frac{z + 2}{\cos(z)}$.

(c) $f(z) = \frac{\sin(z)}{z^2 + 6z + 5}$.

Solution. By Cauchy Integral Theorem, $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ if $f(z)$ is analytic on and between $C_1$ and $C_2$. Hence it is enough to show that $f(z)$ is analytic in $\{|z| \geq 2, |x| + |y| \leq 4\}$.

(a) $f(z)$ is analytic in $\{z \neq \exp(\pm 2\pi i/3)\}$. Since $\exp(\pm 2\pi i/3) \in \{|z| < 2\}$, $f(z)$ is analytic in $\{|z| \geq 2, |x| + |y| \leq 4\}$.

(b) $f(z)$ is analytic in $\{z : \cos(z) \neq 0\} = \{z \neq n\pi + \pi/2\}$. Since $|n\pi + \pi/2| < 2$ for $n = -1, 0$ and $|n\pi + \pi/2| > 4$ for $n \neq -1, 0$ and $n \in \mathbb{Z}$, $f(z)$ is analytic in $\{|z| \geq 2, |x| + |y| \leq 4\}$.

(c) $f(z)$ is analytic in $\{z \neq -1, -5\}$. Since $|-1| < 2$ and $|-5| > 4$, $f(z)$ is analytic in $\{|z| \geq 2, |x| + |y| \leq 4\}$.\]
(8) Let \( f(z) \) and \( g(z) \) be two complex polynomials in \( z \) with \( g(z) \neq 0 \). Show that the integral

\[
\int_{|z|=R} \frac{f(z)}{g(z)} \, dz
\]

is a constant for \( R \) sufficiently large, i.e., there exist \( R_0 > 0 \) and \( C \in \mathbb{C} \) such that

\[
\int_{|z|=R} \frac{f(z)}{g(z)} \, dz = C
\]

for all \( R > R_0 \).

**Proof.** Since \( g(z) \) is a polynomial, it has finitely many complex roots. Let \( z_1, z_2, ..., z_n \) be all the complex roots of \( g(z) \) and let

\[
R_0 = \max(|z_1|, |z_2|, ..., |z_n|).
\]

Then \( f(z)/g(z) \) is holomorphic in \( |z| > R_0 \). For all \( R_2 > R_1 > R_0 \), \( f(z)/g(z) \) is holomorphic in \( R_1 \leq |z| \leq R_2 \). Therefore,

\[
\int_{|z|=R_1} \frac{f(z)}{g(z)} \, dz = \int_{|z|=R_2} \frac{f(z)}{g(z)} \, dz
\]

for all \( R_2 > R_1 > R_0 \), by Cauchy Integral Theorem. It follows that

\[
\int_{|z|=R} \frac{f(z)}{g(z)} \, dz
\]

is a constant for \( R > R_0 \). \( \square \)

(9) In the previous problem, show that

\[
\int_{|z|=R} \frac{f(z)}{g(z)} \, dz = 2\pi i \lim_{z \to \infty} z \frac{f(z)}{g(z)}
\]

for \( R \) sufficiently large, if \( \deg g \geq \deg f + 1 \), where the circle \( |z| = R \) is oriented positively.

**Proof.** By (8),

\[
\int_{|z|=R} \frac{f(z)}{g(z)} \, dz
\]
is a constant for $R$ sufficiently large. Therefore, it suffices to show that

\[
\lim_{R \to \infty} \int_{|z|=R} \frac{f(z)}{g(z)} \, dz = 2\pi i \lim_{z \to \infty} \frac{zf(z)}{g(z)}.
\]

If $\deg g \geq \deg f + 2$, then

\[
\lim_{R \to \infty} \int_{|z|=R} \frac{f(z)}{g(z)} \, dz = 0
\]

and

\[
\lim_{z \to \infty} \frac{zf(z)}{g(z)} = 0.
\]

This proves (0.1) for $\deg g > \deg f + 1$.

Suppose that $\deg g = \deg f + 1$. Let $f(z) = az^n + h_1(z)$ and $g(z) = bz^{n+1} + h_2(z)$, where $n = \deg f$, $\deg h_1 \leq n - 1$ and $\deg h_2 \leq n$. Then

\[
\frac{f(z)}{g(z)} - \frac{a}{b} = \frac{bz h_1(z) - ah_2(z)}{bz g(z)}.
\]

Since $\deg(bz h_1(z) - ah_2(z)) \leq n$ and $\deg(bz g(z)) = n + 2$,

\[
\lim_{R \to \infty} \int_{|z|=R} \frac{bz h_1(z) - ah_2(z)}{bz g(z)} \, dz = 0.
\]

And since

\[
\int_{|z|=R} \frac{a}{bz} \, dz = 2\pi i \left( \frac{a}{b} \right),
\]

we conclude

\[
\lim_{R \to \infty} \int_{|z|=R} \frac{f(z)}{g(z)} \, dz = 2\pi i \left( \frac{a}{b} \right).
\]

Finally, since

\[
\lim_{z \to \infty} \frac{zf(z)}{g(z)} = \lim_{z \to \infty} \frac{az^{n+1} + zh_1(z)}{bz^{n+1} + h_2(z)} = \lim_{z \to \infty} \frac{a + z^{-n}h_1(z)}{b + z^{-n-1}h_2(z)} = \frac{a}{b},
\]

we obtain (0.1) for $\deg g = \deg f + 1$. Thus, (0.1) holds for all $\deg g \geq \deg f + 1$. \qed

(10) Use the conclusion of the previous problem to give a proof of the Fundamental Theorem of Algebra by considering the integral

\[
\int_{|z|=R} \frac{z^{n-1}}{f(z)} \, dz
\]

for a polynomial $f(z)$ of degree $n$. 

Proof. Suppose that \( f(z) \) has no complex roots. Then \( z^{n-1}/f(z) \) is entire and hence
\[
\int_{|z|=R} \frac{z^{n-1}}{f(z)} \, dz = 0.
\]
On the other hand,
\[
\int_{|z|=R} \frac{z^{n-1}}{f(z)} \, dz = 2\pi i \lim_{z \to \infty} \frac{z^n}{f(z)}
\]
for \( R \) sufficiently large by (9). This is a contradiction since
\[
\lim_{z \to \infty} \frac{z^n}{f(z)} = \frac{1}{a_n} \neq 0
\]
for \( f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 \) of degree \( n \). □