Math 311 Assignment #6
Due Nov. 1, 2013

(1) Let $C_R$ denote the upper half of the circle $|z| = R$ for some $R > 1$. Show that
\[ \left| \frac{e^{iz}}{z^2 + z + 1} \right| \leq \frac{1}{(R - 1)^2} \]
for all $z$ lying on $C_R$.

(2) Find the length of the following curves:
   (a) $\gamma(t) = 3t + i$ for $0 \leq t \leq 1$.
   (b) $\gamma(t) = 2i + e^{it}$ for $0 \leq t \leq 1$.
   (c) $\gamma(t) = \sin(t)$ for $-\pi \leq t \leq \pi$.
   (d) $\gamma(t) = (t, t^2)$ for $1 \leq t \leq 2$.

(3) Integrate the following functions over the circle $|z| = 2$, oriented clockwise:
   (a) $z - \overline{z}$.
   (b) $z^2 + 2z + 3$.
   (c) $1/z$.
   (d) $xy$.

(4) Let $\gamma$ be the polygonal path consisting of the line segments $AB$ and $BC$, where $A = 1$, $B = i$ and $C = -1$. Compute the following integrals:
   (a) $\int_{\gamma} zdz$.
   (b) $\int_{\gamma} z^2 dz$.
   (c) $\int_{\gamma} xy dz$.
   (d) $\int_{\gamma} \sin(z) dz$.

(5) Let $f : G \to \mathbb{C}$ be a complex function on an open set $G \subset \mathbb{C}$ and $\gamma : [0, 1] \to G$ be a smooth curve in $G$. Suppose that $f$ has continuous partial derivations $f_x = \partial f / \partial x$ and $f_y = \partial f / \partial y$ in $G$. Then
\[ \frac{d}{dt} (f(\gamma(t))) = f_x(\gamma(t)) \text{Re}(\gamma'(t)) + f_y(\gamma(t)) \text{Im}(\gamma'(t)) \]
for all $t \in [0, 1]$. In particular, if $f$ is holomorphic in $G$, then
\[ \frac{d}{dt} (f(\gamma(t))) = f'(\gamma(t)) \gamma'(t) \]
for all $t \in [0, 1]$. 

(6) Show that
\[ |\log(z)| \leq |\ln |z|| + \pi \]
for all \( z \neq 0 \).

(7) Let \( C_R \) be the circle \(|z| = R \) \((R > 1)\) oriented counterclockwise. Show that
\[ \left| \int_{C_R} \frac{\log(z^2)}{z^2} \, dz \right| < 4\pi \left( \frac{\pi + \ln R}{R} \right) \]
and then
\[ \lim_{R \to \infty} \int_{C_R} \frac{\log(z^2)}{z^2} \, dz = 0. \]

(8) Let \( C_R \) be the circle \(|z| = R \) \((R > 1)\) and \( f(z) \) and \( g(z) \) be two complex polynomials of degrees \( m \) and \( n \), respectively. If \( n - m \geq 2 \), then
\[ \lim_{R \to \infty} \int_{C_R} \frac{f(z)}{g(z)} \, dz = 0. \]

(9) A set \( G \subset \mathbb{C} \) is called star-shaped if there is a point \( p \in G \) such that the line segment \( pq \subset G \) for all \( q \in G \). Show that if \( G \) is star-shaped, every two closed curves in \( G \) are \( G \)-homotopic.

(10) Let \( G = \mathbb{C}^* \). Show that \( \gamma_1 \) and \( \gamma_2 \) are \( G \)-homotopic for
(a) \( \gamma_1 = \{|z| = 1\} \) and \( \gamma_2 \) is the boundary of the square \(|x| \leq 1, |y| \leq 1\), both oriented counterclockwise.
(b) \( \gamma_1 = \{|z - 2| = 1\} \) and \( \gamma_2 = \{|z + 2| = 1\} \), both oriented counterclockwise.