Solutions for Math 311 Assignment #10

(1) Prove the complex L’Hospital’s rule: If $f(z)$ and $g(z)$ are analytic at $z_0$ and $f(z_0) = g(z_0) = 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}$$

Proof. Let $m$ and $n$ be the multiplicities of the zeros of $f(z)$ and $g(z)$ at $z_0$. Then $f(z) = (z - z_0)^m a(z)$ and $g(z) = (z - z_0)^n b(z)$ for some positive integers $m$ and $n$ and some functions $a(z)$ and $b(z)$ analytic at $z_0$ and satisfying $a(z_0) \neq 0$ and $b(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m a(z)}{(z - z_0)^n b(z)} = \begin{cases} 0 & \text{if } m > n \\ \frac{a(z_0)}{b(z_0)} & \text{if } m = n \\ \infty & \text{if } m < n \end{cases}$$

On the other hand,

$$\lim_{z \to z_0} \frac{f'(z)}{g'(z)} = \lim_{z \to z_0} \frac{(z - z_0)^{m-1}((z - z_0)a'(z) + a(z))}{(z - z_0)^{n-1}((z - z_0)b'(z) + b(z))} = \begin{cases} 0 & \text{if } m > n \\ \frac{a(z_0)}{b(z_0)} & \text{if } m = n \\ \infty & \text{if } m < n \end{cases}$$

Therefore,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}.$$ 

□

(2) Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e} \left( \sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2} \right)$$

for $0 < |z+1| < \infty$.

Proof. Since

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

for $|z| < \infty$, we have

$$e^{z+1} = \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!}$$

\[\text{http://www.math.ualberta.ca/~xichen/math31113f/hw10sol.pdf}\]
for \(|z+1| < \infty\) by substituting \(z+1\) for \(z\). Therefore,

\[
\frac{e^z}{(z+1)^2} = \frac{e^{z+1}}{e(z+1)^2} = \sum_{n=0}^{\infty} \frac{(z+1)^{n-2}}{e(n!)}
\]

\[
= \frac{1}{e} \left( \frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=2}^{\infty} \frac{(z+1)^{n-2}}{n!} \right)
\]

\[
= \frac{1}{e} \left( \frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=0}^{\infty} \frac{(z+1)^{n}}{(n+2)!} \right)
\]

for \(0 < |z+1| < \infty\). \(\Box\)

(3) Give two Laurent Series expansions in powers of \(z\) for the function

\[
f(z) = \frac{1}{z^2(1-z)}
\]

and specify the regions in which those expansions are valid.

Solution. We observe that \(f(z)\) is analytic in \(\{z \neq 0,1\}\). So it is analytic in \(0 < |z| < 1\) and \(1 < |z|\). When \(0 < |z| < 1\),

\[
\frac{1}{z^2(1-z)} = \frac{1}{z^2} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2}
\]

\[
= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=2}^{\infty} z^{n-2} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n
\]

When \(1 < |z| < \infty\),

\[
\frac{1}{z^2(1-z)} = -\frac{1}{z} \frac{1}{z+1} = -\frac{1}{z^3} \sum_{n=0}^{\infty} z^{-n}
\]

\[
= -\sum_{n=0}^{\infty} \frac{1}{z^{n+3}} = \sum_{n=3}^{\infty} \frac{1}{z^n}
\]

\(\Box\)

(4) For each of the following complex functions, do the following:

- find all its singularities in \(\mathbb{C}\);
- write the principal part of the function at each singularity;
- for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
- compute the residue of the function at each singularity.
(a) \( \frac{1}{z + z^2} \);
(b) \( z \cos \left( \frac{1}{z} \right) \);
(c) \( \frac{\sinh z}{z^4(1 - z^2)} \).

**Solution.** (a) It has two singularities at 0 and \(-1\).

At \( z = 0 \), since

\[
\frac{1}{z + z^2} = \frac{1}{z(1 + z)} = \frac{1}{z} \left( \frac{1}{1 + z} \right)_{z=0} + \sum_{n=1}^{\infty} a_n z^n
\]

\[
= \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{n-1}
\]

it has a pole of order 1 at 0 with principal part

\[
\frac{1}{z}
\]

and residue

\[
\text{Res}_{z=0} \frac{1}{z + z^2} = 1.
\]

At \( z = -1 \), since

\[
\frac{1}{z + z^2} = \frac{1}{z + (-1)^2} = \frac{1}{z + 1} \left( \frac{1}{z} \right)_{z=-1} + \sum_{n=1}^{\infty} a_n (z + 1)^n
\]

\[
= -\frac{1}{z + 1} + \sum_{n=1}^{\infty} a_n (z + 1)^{n-1}
\]

it has a pole of order 1 at \(-1\) with principal part

\[
-\frac{1}{z + 1}
\]

and residue

\[
\text{Res}_{z=-1} \frac{1}{z + z^2} = -1.
\]

(b) It has a singularity at 0.
Since

\[ z \cos \left( \frac{1}{z} \right) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n-1}} = z - \frac{1}{2z} + \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)! z^{2n-1}}, \]

it has an essential singularity at 0 with principal part

\[ -\frac{1}{2z} + \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)! z^{2n-1}}, \]

and residue

\[ \text{Res}_{z=0} z \cos \left( \frac{1}{z} \right) = -\frac{1}{2}. \]

(c) It has three singularities at 0 and ±1.

At \( z = 0 \), since

\[
\frac{\sinh z}{z^4(1 - z^2)} = \frac{e^z - e^{-z}}{2z^4(1 - z^2)} = \frac{1}{2z^4} \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!} \right) \left( \sum_{n=0}^{\infty} z^{2n} \right)
\]

\[
= \frac{1}{2z^4} \left( \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{(2n + 1)!} \right) \left( \sum_{n=0}^{\infty} z^{2n} \right)
\]

\[
= \frac{1}{z^3} \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n + 1)!} \right) \left( \sum_{n=0}^{\infty} z^{2n} \right)
\]

\[
= \frac{1}{z^3} \left( 1 + \frac{x^2}{6} + \ldots \right) \left( 1 + z^2 + \ldots \right)
\]

\[
= \frac{1}{z^3} + \left( 1 + \frac{1}{6} \right) \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n
\]

it has a pole of order 3 at 0 with principal part

\[ \frac{1}{z^3} + \frac{7}{6z} \]

and residue

\[ \text{Res}_{z=0} \frac{\sinh z}{z^4(1 - z^2)} = \frac{7}{6}. \]
At \( z = 1 \), since
\[
\frac{\sinh z}{z^4(1 - z^2)} = \frac{1}{1 - z} \frac{\sinh z}{z^4(1 + z)} =
\]
\[
= -\frac{\sinh(1)}{2} \frac{1}{z - 1} - \sum_{n=1}^\infty a_n (z - 1)^{n-1}
\]
it has a pole of order 1 at 1 with principal part
\[
-\frac{\sinh(1)}{2} \frac{1}{z - 1}
\]
and residue
\[
\text{Res}_{z=1} \frac{\sinh z}{z^4(1 - z^2)} = -\frac{\sinh(1)}{2}.
\]

At \( z = -1 \), since
\[
\frac{\sinh z}{z^4(1 - z^2)} = \frac{1}{z + 1} \frac{\sinh z}{z^4(1 - z)} =
\]
\[
= -\frac{\sinh(1)}{2} \frac{1}{z + 1} + \sum_{n=1}^\infty a_n (z + 1)^{n-1}
\]
it has a pole of order 1 at \(-1\) with principal part
\[
-\frac{\sinh(1)}{2} \frac{1}{z + 1}
\]
and residue
\[
\text{Res}_{z=-1} \frac{\sinh z}{z^4(1 - z^2)} = -\frac{\sinh(1)}{2}.
\]

(5) Let \( f(z) \) and \( g(z) \) be two holomorphic functions in a connected open set \( G \). Show that if \( f^{(n)}(z_0) = g^{(n)}(z_0) \) for some \( z_0 \in G \) and all \( n \geq 0 \), then \( f(z) \equiv g(z) \) in \( G \).
Proof. Since \( f(z) \) and \( g(z) \) are holomorphic in \( G \), \( f(z) \) and \( g(z) \) are holomorphic in \(|z - z_0| < r\) for some \( r > 0\). Therefore,

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
\]
and

\[
g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^n
\]
in \(|z - z_0| < r\). And since \( f^{(n)}(z_0) = g^{(n)}(z_0) \) for all \( n \), \( f(z) \equiv g(z) \) in \(|z - z_0| < r\) \( \subset G \). Therefore, \( f(z) \equiv g(z) \) in \( G \) since \( G \) is connected. \( \square \)

(6) Evaluate the integrals of the following functions around the circle \(|z| = 3\) oriented counterclockwise:

(a) \( \frac{\exp(-z)}{z^2} \);

(b) \( \frac{z + 1}{z^2 - 2z} \).

Solution. (a) By Cauchy Integral Theorem,

\[
\int_{|z|=3} \frac{\exp(-z)}{z^2} \, dz = \int_{|z|=r} \frac{\exp(-z)}{z^2} \, dz = 2\pi i \text{Res}_{z=0} \frac{\exp(-z)}{z^2}.
\]

And since

\[
\frac{\exp(-z)}{z^2} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n-2}}{n!}
\]

\( \text{Res}_{z=0} \exp(-z)/z^2 = -1 \) and hence

\[
\int_{|z|=3} \frac{\exp(-z)}{z^2} \, dz = -2\pi i.
\]

(b) By Cauchy Integral Theorem,

\[
\int_{|z|=3} \frac{z + 1}{z^2 - 2z} \, dz = \int_{|z|=r} \frac{z + 1}{z^2 - 2z} \, dz + \int_{|z-2|=r} \frac{z + 1}{z^2 - 2z} \, dz
\]

\[
= 2\pi i \text{Res}_{z=0} \frac{z + 1}{z^2 - 2z} + 2\pi i \text{Res}_{z=2} \frac{z + 1}{z^2 - 2z}.
\]

And since

\[
\frac{z + 1}{z^2 - 2z} = -\frac{1}{2z} + \frac{3}{2(z - 2)}
\]
\[
\text{Res}_{z=0} \frac{z+1}{z^2-2z} = \text{Res}_{z=0} \left( -\frac{1}{2z} \right) + \text{Res}_{z=0} \frac{3}{2(z-2)} \\
= \text{Res}_{z=0} \left( -\frac{1}{2z} \right) = -\frac{1}{2} \\
\]

and

\[
\text{Res}_{z=2} \frac{z+1}{z^2-2z} = \text{Res}_{z=2} \left( -\frac{1}{2z} \right) + \text{Res}_{z=2} \frac{3}{2(z-2)} \\
= \text{Res}_{z=2} \left( \frac{3}{2(z-2)} \right) = \frac{3}{2}.
\]

Therefore,

\[
\int_{|z|=3} \frac{z+1}{z^2-2z} \, dz = 2\pi i \left( \frac{3}{2} - \frac{1}{2} \right) = 2\pi i.
\]

\(\square\)

(7) Find the Laurent series of

\[
\frac{1}{e^z - 1}
\]

in 0 < |z| < R up to \(z^6\) and show that the series converges in 0 < |z| < \(\sqrt{2\pi}\).

\textbf{Solution.} Let \(f(z) = 1/(e^z - 1)\). Since

\[
e^z - 1 = \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}
\]

\(e^z - 1\) has a zero at 0 of multiplicity one and hence \(f(z)\) has pole at 0 of order 1. So the Laurent series of \(f(z)\) is given by

\[
f(z) = \sum_{n=-1}^{\infty} a_n z^n = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \sum_{n \geq 4} a_n z^n
\]

in 0 < |z| < \(r\) for some \(r > 0\).

Since \((e^z - 1)f(z) = 1\), we have

\[
1 = \left( a_{-1} + a_0 z + a_1 z^2 + a_2 z^3 + a_3 z^4 + \sum_{n \geq 5} a_n z^n \right) \\
\left( 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \sum_{n \geq 5} \frac{z^n}{(n+1)!} \right).
\]
Comparing the coefficients of $1$, $z$, $z^2$, $z^3$ and $z^4$ on both sides, we obtain

\[
\begin{aligned}
a_{-1} &= 1 \\
a_0 + \frac{a_{-1}}{2} &= 0 \\
a_1 + \frac{a_0}{2} + \frac{a_{-1}}{6} &= 0 \\
a_2 + \frac{a_1}{2} + \frac{a_0}{6} + \frac{a_{-1}}{24} &= 0 \\
a_3 + \frac{a_2}{2} + \frac{a_1}{6} + \frac{a_0}{24} + \frac{a_{-1}}{120} &= 0
\end{aligned}
\]

Solving it, we have $a_{-1} = 1$, $a_0 = -1/2$, $a_1 = 1/12$, $a_2 = 0$ and $a_3 = -1/720$. Hence

\[
f(z) = \frac{1}{z} - \frac{1}{2} + \frac{z^3}{12} - \frac{z^6}{720} + \sum_{n \geq 4} a_n z^n
\]

and

\[
\frac{1}{e^{z^2} - 1} = f(z^2) = \frac{1}{z^2} - \frac{1}{2} + \frac{z^2}{12} - \frac{z^6}{720} + \sum_{n \geq 4} a_n z^{2n}.
\]

Note that $f(z)$ is analytic in $\{ \{ z : e^z - 1 \neq 0 \} = \{ z \neq 2n\pi i \}$. So it is analytic in $0 < |z| < 2\pi$. Therefore, $f(z^2)$ is analytic in $0 < |z^2| < 2\pi$, i.e., $0 < |z| < \sqrt{2\pi}$. So the series converges in $0 < |z| < \sqrt{2\pi}$. □

(8) Compute the contour integral

\[
\int_C \frac{z^{2013}}{z^{2013} + z^{2012} + z^{2010} + 1} \, dz,
\]

where $C$ is the circle $|z| = 2$ oriented counter-clockwise.

Solution. First, we prove that all zeroes of $z^{2013} + z^{2012} + z^{2010} + 1$ lie inside the circle $|z| = 2$. Otherwise, $z_0^{2013} + z_0^{2012} + z_0^{2010} + 1 = 0$ for some $|z_0| \geq 2$. Then

\[
z_0^{2013} + z_0^{2012} + z_0^{2010} + 1 = 0 \Rightarrow 1 + \frac{1}{z_0} + \frac{1}{z_0^3} + \frac{1}{z_0^{2013}} = 0
\]

On the other hand,

\[
\left| 1 + \frac{1}{z_0} + \frac{1}{z_0^3} + \frac{1}{z_0^{2013}} \right| \geq 1 - \frac{1}{|z_0|} - \frac{1}{|z_0|^3} - \frac{1}{|z_0|^{2013}} \\
\geq 1 - \frac{1}{2} - \frac{1}{2^3} - \frac{1}{2^{2013}} > 0
\]
for $|z_0| \geq 2$. Contradiction.

So all zeroes of $z^{2013} + z^{2012} + z^{2010} + 1$ lie inside the circle $|z| = 2$ and hence $z^{2013}/(z^{2013} + z^{2012} + z^{2010} + 1)$ is analytic in $|z| \geq 2$. Therefore,

$$
\int_{|z|=2} \frac{z^{2013}}{z^{2013} + z^{2012} + z^{2010} + 1} \, dz = \int_{|z|=R} \frac{z^{2013}}{z^{2013} + z^{2012} + z^{2010} + 1} \, dz
$$

for all $R > 2$ by CIT.

We observe that

$$
\frac{z^{2013}}{z^{2013} + z^{2012} + z^{2010} + 1} - \frac{z}{z + 1} = -\frac{z(z^{2010} + 1)}{(z + 1)(z^{2013} + z^{2012} + z^{2010} + 1)}.
$$

We have proved that

$$
\int_{|z|=R} \frac{z(z^{2010} + 1)}{(z + 1)(z^{2013} + z^{2012} + z^{2010} + 1)} \, dz = 0
$$

for $R$ sufficiently large. Therefore,

$$
\int_{|z|=2} \frac{z^{2013}}{z^{2013} + z^{2012} + z^{2010} + 1} \, dz = \int_{|z|=R} \frac{z^{2013}}{z^{2013} + z^{2012} + z^{2010} + 1} \, dz = \int_{|z|=R} \frac{z}{z + 1} \, dz = -2\pi i.
$$

(9) Evaluate the contour integral of the following functions around the circle $|z| = 2013$ oriented counterclockwise:

(a) $\frac{1}{\sin z}$;

(b) $\frac{1}{e^{2z} - e^z}$.

Solution.  (a) $f(z) = 1/\sin z$ is analytic in $\{z \neq n\pi : n \in \mathbb{Z}\}$. It has a pole of order one at $n\pi$ since $(\sin z)|_{z=n\pi} = \cos(n\pi) = (-1)^n \neq 0$. So

$$
\text{Res}_{z=n\pi} \frac{1}{\sin z} = \frac{1}{\cos(n\pi)} = (-1)^n.
$$
Therefore,
\[ \int_{|z| = 2013} \frac{dz}{\sin z} = 2\pi i \sum_{|n\pi| < 2013} \text{Res}_{z=n\pi} \frac{1}{\sin z} = 2\pi i \sum_{|n| \leq 640} (-1)^n = 2\pi i. \]

(b) \( f(z) = 1/(e^{2z} - e^z) \) is analytic in \( \{ e^{2z} - e^z \neq 0 \} = \{ e^z \neq 1 \} = \{ z \neq 2n\pi i : n \in \mathbb{Z} \}. \)

Since \( (e^{2z} - e^z)'_{|z=2n\pi i} = 1 \neq 0 \), \( f(z) \) has a pole of order one at \( 2n\pi i \). So
\[ \text{Res}_{z=2n\pi i} \frac{1}{e^{2z} - e^z} = \frac{1}{2e^{2z} - e^z} \bigg|_{z=2n\pi i} = 1. \]

Therefore,
\[ \int_{|z| = 2013} \frac{dz}{e^{2z} - e^z} = 2\pi i \sum_{|2n\pi i| < 2013} \text{Res}_{z=2n\pi i} \frac{1}{e^{2z} - e^z} = 2\pi i \sum_{|2n\pi i| < 2013} 1 = 2\pi i \sum_{|n| \leq 320} 1 = 1282\pi i. \]

(10) Let \( f(z) \) and \( g(z) \) be two complex polynomials in \( z \). Show that if \( \deg f \leq \deg g - 2 \), the sum of the residues of \( f(z)/g(z) \) at the zeros of \( g(z) \) is zero.

**Proof.** Let \( z_1, z_2, \ldots, z_n \) be the zeros of \( g(z) \) and let \( R_0 = \max(|z_1|, |z_2|, \ldots, |z_n|) \).

By Residue theorem, we have
\[ \int_{|z|=R} \frac{f(z)}{g(z)} \, dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} \frac{f(z)}{g(z)} \]

for all \( R > R_0 \). Since \( \deg f \leq \deg g - 2 \),
\[ \lim_{R \to \infty} \int_{|z|=R} \frac{f(z)}{g(z)} \, dz = 0 \]

and hence
\[ \sum_{k=1}^{n} \text{Res}_{z=z_k} \frac{f(z)}{g(z)} = 0. \]