Math 311 Final Review

Some information on the final:

- Time and location: 9-12, Dec. 16, 2013, CCIS 1-160
- Sections covered by the final: (BMPS) 1.0-1.5, 2.1-2.4, 3.1-3.2, 3.4-3.5, 4.1-4.3, 5.1-5.3, 7.1-7.4, 8.1-8.3, 9.1-9.2

Some review problems for the final:

1. Let \( f(z) \) be the principal branch of \( z^{-i} \).
   
   (a) Find \( f(i) \).
   
   (b) Show that \( f(z_1) f(z_2) = \lambda f(z_1 z_2) \) for all \( z_1, z_2 \neq 0 \), where \( \lambda = 1, e^{2\pi} \) or \( e^{-2\pi} \).

2. Let \( f(z) = \frac{z^2}{z^2 - 3z + 2} \). Find the Laurent series of \( f(z) \) in each of the following domains:
   
   (a) \( 1 < |z| < 2 \)
   
   (b) \( 1 < |z - 3| < 2 \)

3. Compute the integral
   
   \[
   \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2} \, dx.
   \]

4. Compute the integral
   
   \[
   \int_{0}^{\pi} \frac{d\theta}{2 - \cos \theta}.
   \]

5. For each of the following complex functions, do the following:
   
   - find all its singularities in \( \mathbb{C} \);
   
   - write the principal part of the function at each singularity;
   
   - for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
   
   - compute the residue of the function at each singularity.

   (a) \( f(z) = \tan z \)

   (b) \( f(z) = (1 - z^2) \sin \left( \frac{1}{z} \right) \)

   (c) \( f(z) = \frac{e^z}{z^{2011}} \cos \frac{z}{z^2} \)

   (d) \( f(z) = \frac{\cos z}{z^2 - z^3} \)
(6) Let \( f(z) = u(x, y) + iv(x, y) \) be an entire function with the property that \( v(x, y) \geq x \) for all \( z = x + yi \), where \( u(x, y) = \text{Re}(f(z)) \) and \( v(x, y) = \text{Im}(f(z)) \). Show that \( f(z) \) is a polynomial of degree 1.

(7) Do the following:
   
   (a) Find \( \sin \left( \frac{\pi}{4} + i \right) \).
   
   (b) Show that \( |\sin z|^2 = (\sin x)^2 + (\sinh y)^2 \) for all complex numbers \( z = x + yi \).
   
   (c) Let \( C_N \) be the boundary of the square
   \[
   \left\{ |x| \leq N\pi + \frac{\pi}{2}, |y| \leq N\pi + \frac{\pi}{2} \right\}
   \]
   oriented counterclockwise, where \( N \) is a positive integer. Show that
   \[
   \lim_{N \to \infty} \int_{C_N} \frac{dz}{z^2 \sin z} = 0.
   \]
   
   (d) Use Cauchy Integral Theorem or Residue Theorem to show that
   \[
   \frac{1}{2\pi i} \int_{C_N} \frac{dz}{z^2 \sin z} = \frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2}
   \]
   and conclude that
   \[
   \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots
   \]

(8) Let \( f(z) \) be an entire function satisfying \( |f(z)| \leq |z|^2 \) for all \( z \). Show that \( f(z) \equiv az^2 \) for some complex constant \( a \) satisfying \( |a| \leq 1 \).

(9) Compute the integral
\[
\int_0^{\infty} \frac{x}{x^4 + 1} \, dx.
\]

(10) Compute the contour integral
\[
\int_C \frac{z^{2012}}{z^{2013} + z^2 + z + 1} \, dz
\]
where \( C \) is the circle \( |z| = 2 \) oriented counter-clockwise.