(1) No books, notes or calculators are allowed.
(2) Show your work in details.

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(1) (25 pts) Do the following:

(a) (5 pts) Compute $\sin\left(\frac{\pi}{3} + i\right)$

(b) (10 pts) Show that

\[ \sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \]

for all complex numbers $z_1$ and $z_2$.

(c) (10 pts) Show that $\sin(\bar{z})$ is nowhere holomorphic in $\mathbb{C}$.
(2) (30 pts) Let

\[ f(z) = \frac{1}{z^2 - z^3} \]

Find the Laurent series of \( f(z) \) in each of the following domains:

(a) (15 pts) \(|z| > 1\)

(b) (15 pts) \(0 < |z - 1| < 1\)
(3) (25 pts) Compute the integral
\[ \int_{-\infty}^{\infty} \frac{\sin(tx)}{x^2 + 2x + 2} \, dx \]
for \( t < 0 \).
(4) (20 pts) Evaluate the contour integral of the following functions around the circle $|z| = 100$ oriented counterclockwise:

(a) $\frac{1}{\sin(z)}$

(b) $\frac{1}{e^{2z} - e^z}$
(5) (40 pts) For each of the following complex functions, do the following:

- find all its singularities in \( \mathbb{C} \);
- write down the principal part of the function at each singularity;
- for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity; if it is a pole, determine its order;
- compute the residue of the function at each singularity.

(a) \( \frac{1}{z + z^2} \)

(b) \( z \cos \left( \frac{1}{z} \right) \)
(c) \( f(z) = \frac{\sin(z)}{z^{2013}} \)

(d) \( \frac{\sinh z}{z^4(1 - z^2)} \)
(6) (30 pts) Let $f(z) = u(x, y) + iv(x, y)$ be an entire function satisfying
\[ u(x, y) + v(x, y) \geq 0 \]
for all $z = x + yi$, where $u(x, y) = \text{Re}(f(z))$ and $v(x, y) = \text{Im}(f(z))$. Show that $f(z)$ is a constant.
(7) (30 pts) Let $f(z)$ be an entire function satisfying
\[ |f(z)| \leq |z^2 + z + 1| \]
for all $z$. Show that $f(z) \equiv a(z^2 + z + 1)$ for some complex constant $a$ satisfying $|a| \leq 1$. 