Uniqueness of Taylor Series

Thm. If \( f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \)
in \( \{ |z - z_0| < r \} \) for some \( r > 0 \),
then \( a_n = \frac{f^{(n)}(z_0)}{n!} \)

Proof. \( \frac{d^n}{dz^n} \) on both sides

\[ \Rightarrow f^{(n)}(z) = \sum_{m=0}^{\infty} a_m \left( (z - z_0)^m \right)^{(n)} \]

\[ = \sum_{m=n}^{\infty} m(m-1) \cdots (m-n+1) a_m (z - z_0)^{m-n} \]
in \( \{ |z - z_0| < r \} \) \( \Rightarrow f^{(n)}(z_0) = n! \, a_n \)
— Zeros of Holomorphic Functions

Defn. Let \( f(z) \) be a holomorphic function at \( z_0 \). We say that \( f(z) \)
has a zero at \( z_0 \) of multiplicity \( m \) if
\[
f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n
\]
in \( |z-z_0| < r \) for some \( a_m \neq 0 \), or equivalently
\[
f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0
\]
and \( f^{(m)}(z_0) \neq 0 \)

E.g. Let \( f(z) = (z-z_1)^{m_1} (z-z_2)^{m_2} \ldots (z-z_n)^{m_n} \)
be a polynomial in \( z \) with \( z_1, z_2, \ldots, z_n \) distinct roots.
\( f(z) \) has a zero at \( z_k \) of multiplicity \( m_k \).

e.g. Find zeros of \( \sin z \) and their multiplicities:
\[
z = n\pi \quad f'(n\pi) = \cos(n\pi) \neq 0
\]
\( \Rightarrow \sin z \) has zeros at \( n\pi \) of multiplicities 1.

e.g. Find zeros of \( 1 - \cos z \) and their multiplicities:
\[
1 - \cos z = 0 \iff z = 2n\pi
\]
\( f'(2n\pi) = \sin(2n\pi) = 0 \)
\( f''(2n\pi) = \cos(2n\pi) \neq 0 \)
l - cos z has zeros at \(2n\pi\) of multiplicities 2

Thm. Let \(f(z)\) be a holomorphic function in \(\{ |z-z_0| < r \}\). Then \(f(z)\) has a zero at \(z_0\) of multiplicity \(m\) iff \(f(z) = (z-z_0)^m g(z)\) for some \(g(z)\) holomorphic in \(\{ |z-z_0| < r \}\) and \(g(z_0) \neq 0\).

Proof. \(f(z)\) has a zero at \(z_0\) of multiplicity \(m\) \(\Rightarrow\)

\[
f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n \quad \text{in} \quad |z-z_0| < r
\]

\(a_m \neq 0\)
\[ f(z) = (z - z_0)^m \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m} \]

Let \( g(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m} \)

\[ \sum_{n=m}^{\infty} a_n (z - z_0)^{n} \] converges in \(|z - z_0| < r\)

\[ \Rightarrow \lim_{n \to \infty} |a_n s^n| = 0 \text{ for all } |s| < r \]

\[ \Rightarrow \lim_{n \to \infty} |a_n s^{n-m}| = 0 \text{ for all } |s| < r \]

\[ \Rightarrow \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m} \] converges in \(|z - z_0| < r\)

\[ \Rightarrow g(z) \text{ is holomorphic in } |z - z_0| < r \]

Also \( g(z_0) = a_m \neq 0 \)
\[ f(z) = (z - z_0)^m g(z) \] for some \( g(z) \) holomorphic in \( |z - z_0| < r \) and \( g(z_0) \neq 0 \)

Let \( g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^n \)

\[ = \sum_{n=0}^{\infty} b_n (z - z_0)^n \]

\( b_0 = g(z_0) \neq 0 \)

Then \( f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^{m+n} \)

\[ = \sum_{n=m}^{\infty} b_{n-m} (z - z_0)^n \]

\( b_0 \neq 0 \Rightarrow f(z) \) has a zero at \( z_0 \) of multiplicity \( m \)
Thm. (Complex L'Hospital)
If \( f(z) \) and \( g(z) \) are analytic at \( z_0 \) and \( f(z_0) = g(z_0) = 0 \), then
\[
\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}
\]

Thm. (Zeros of Holomorphic Functions are isolated)
If \( f(z) \) is analytic at \( z_0 \) and \( f(z_0) = 0 \), then
— either \( f(z) = 0 \) in \([z - z_0] < r\)
— or \( f(z) \neq 0 \) in \([0 < |z - z_0| < r]\)
for some \( r > 0 \).
Proof. \( f(z) \) is analytic at \( z_0 \)
\[ f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \]
in \( |z-z_0| < r \)

If \( a_n = 0 \) for all \( n \geq 0 \)
then \( f(z) \equiv 0 \) in \( |z-z_0| < r \)

If \( a_n \neq 0 \) for some \( n \),
let \( m \) be the smallest \( m \)
\( s.t. \ a_m \neq 0 \), i.e., \( a_0 = a_1 = \ldots = a_{m-1} = 0 \) and \( a_m \neq 0 \)

\( \Leftrightarrow f(z) \) has a zero at \( z_0 \)
of multiplicity \( m \)
\[ f(z) = (z - z_0)^m \cdot g(z) \]

for \( g(z) \) analytic at \( z_0 \) and \( g(z_0) \neq 0 \)

Let \( M = |g(z_0)| \)

\( g(z) \) analytic at \( z_0 \)

\( g(z) \) continuous at \( z_0 \)

\[ \lim_{z \to z_0} g(z) = g(z_0) \]

\( \Rightarrow \) There is \( r' > 0 \) s.t.

\[ |g(z) - g(z_0)| < \frac{M}{2} \quad \text{for} \quad |z - z_0| < r' \]

\( \Rightarrow \) \[ |g(z)| \geq |g(z_0)| - |g(z_0) - g(z)| \]

\[ > M - \frac{M}{2} = \frac{M}{2} \]
\[ \Rightarrow g(z) \neq 0 \quad \text{in} \quad |z-z_0| < r' \]
\[ \Rightarrow f(z) = (z-z_0)^m g(z) \neq 0 \quad \text{in} \quad \{0 < |z-z_0| < r'\} \]

Cor. If \( f(z) \) is analytic at \( z_0 \) and there is a sequence \( \{z_1, z_2, \ldots, z_n, \ldots\} \) such that \( z_n \neq z_0 \), \( \lim_{n \to \infty} z_n = z_0 \), and \( f(z_n) = 0 \), then \( f(z) \neq 0 \) in \( \{1 < |z-z_0| < r'\} \) for some \( r > 0 \). In other words, the zeros of \( f(z) \) are isolated unless \( f(z) \neq 0 \).