Cauchy Integral Theorem (CIT) Let $f(z)$ be a holomorphic function in an open set $G \subset \mathbb{C}$. If $\gamma_1$ and $\gamma_2$ are two piecewise smooth closed curves in $G$ that are $G$-homotopic to each other, then

$$\int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz$$

In particular, $\gamma \sim \{1 \text{ pt}\}$

$$\int_{\gamma} f(z) \, dz = 0$$

e.g. If $G$ is simply connected,

$$\int_{\gamma} f(z) \, dz = 0$$
for all holomorphic functions $f(z)$ in $G$ and all piece-wise smooth closed curves $\gamma$.

e.g. $f(z)$ entire $\Rightarrow \int_{\gamma} f(z) \, dz = 0$

for all piece-wise smooth closed curves $\gamma$. $\iff \int_{\gamma} f(z) \, dz = \int_{z_0}^{z_1} f(z) \, dz$

independent of the choice of a curve from $z_0$ and $z_1$.

e.g. $G$ is convex $\Rightarrow \int_{\gamma} f(z) \, dz = 0$

for all $f(z)$ holomorphic in $G$ and $\gamma$ closed and piece-wise smooth in $G$. 
e.g. $G$ is star-shaped $\iff$

There is $p \in G$ s.t. $\overline{T}$ for all $q \in G \Rightarrow G$ is simply-connected

$\Rightarrow \int_Y f(z) \, dz = 0$ for all $f(z)$ holomorphic in $G$ and $Y$ piece-wise smooth and closed in $G$

Proof. Since $Y_1 \sim_G Y_2$ and $Y_2 \sim_G Y_3$

$\Rightarrow Y_1 \sim_G Y_3$, it suffices to show that $Y \sim_G \{p\}$ for all $Y$ closed.

$h: [0,1] \times [0,1] \to G$ given by

$h(t,s) = (1-s)Y(t) + sP$

Verify that $h$ is a continuous function from $[0,1] \times [0,1]$ to $G$. 

\[ h(t,0) = y(t), \quad h(t,1) = p \]
\[ h(0,s) = h(1,s) \quad \text{for} \quad 0 \leq s \leq 1 \]

**Example:**
\[
\int_{|z-100|=100} e^{\sin z} \, dz = 0
\]
since \( e^{\sin z} \) is entire.

**Example:**
\[
\int_{|z-2|=1} \frac{e^z}{z^2+1} \, dz = 0
\]
since \( \frac{e^z}{z^2+1} \) is holomorphic everywhere in \( \{ |z-2| \leq 1 \} = G \)
and \( G \) is convex and hence simply connected.

**Example:**
Show that \( \int_{Y_1} \frac{1}{z} \, dz = \int_{Y_2} \frac{1}{z} \, dz \).
for $\gamma_1 = \{ |z| = 1 \}$ and $\gamma_2$ the boundary of $\{ |x| \leq 1, |y| \leq 1 \}$

Proof. Want to show that

$$\gamma_1 \sim_{C^*} \gamma_2.$$ 

$$\gamma_2(t) = \begin{cases} 
(1 - 4t)z_0 + 4tz_1 & \text{if } 0 \leq t \leq \frac{1}{4} \\
(4t + 2)z_1 + (4t - 1)z_2 & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\
(-4t + 3)z_2 + (4t - 2)z_3 & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\
(-4t + 4)z_3 + (4t - 3)z_0 & \text{if } \frac{3}{4} \leq t \leq 1.
\end{cases}$$

$$\gamma_1(t) = e^{2\pi i t},$$ 

$$\gamma_1(t) \rightarrow \gamma_2(t).$$

Let $h(t, s) = (1 - s) \gamma_1 (t) + s \gamma_2(t)$

Verify that $h : [0, 1] \times [0, 1] \rightarrow C^*$

$0 \neq h([0,1] \times [0,1])$
Jordan Curve Theorem

Thm. Let \( Y : [0, 1] \to C \) be a simply closed curve (\( Y(t_1) = Y(t_2) \) for \( 0 \leq t_1 < t_2 \leq 1 \) if \( t_1 = 0 \) and \( t_2 = 1 \)). Then \( C \setminus Y = G_1 \cup G_2 \)

where

- \( G_1 \cap G_2 = \emptyset \)

- \( G_1 \) and \( G_2 \) are open and connected

- \( G_1 \) is bounded and \( G_2 \) is unbounded

- \( G_1 \) is simply connected