Integration of Complex Functions

\[ \int \int_D f(x,y) \, dx \, dy \]

\[ \int \int \left( \int f(x,y) \, dx + g(x,y) \, dy \right) \]

Line Integral

Defn. \( G \subset \mathbb{C} \) A curve \( \gamma \) in \( G \) is a continuous function \( \gamma : [a,b] \rightarrow G \). Sometimes we allow \( \gamma : (a,b) \rightarrow G \) continuously.

\( \gamma \) is smooth if \( \gamma(t) \) is differentiable on \([a,b]\) \( \gamma(t) = x(t) + i \, y(t) \)

\( x'(t) \) and \( y'(t) \) exist and are continuous...
$\gamma$ is piecewise smooth if

$\gamma(t)$ is differentiable on $[a, c_i], [c_i, c_{i+1}], \ldots, [c_n, b]$ continuously.

e.g. A circle oriented counterclockwise

$\gamma : [0, 1] \rightarrow \mathbb{C}$

$\gamma(t) = z_0 + re^{2\pi it}$

clockwise $\gamma(t) = z_0 + re^{-2\pi it}$

$\gamma : [0, 1] \rightarrow \mathbb{C}$ a curve

$\gamma(1-t)$ and $\gamma(t)$ are oppositely oriented.

e.g. The line segment from $z_1$ to $z_2$

$\gamma : [0, 1] \rightarrow \mathbb{C}$
\[ \gamma(t) = (1-t)z_1 + tz_2. \]

e.g. Parameterize the boundary of the polygon \[ \{ |x+y| \leq 1, \quad |x-y| \leq 1 \} \] oriented counterclockwise.

\[ \gamma(t) = \begin{cases} (1-t)+ti & [0,1] \\
(2-t)i+(t-i)[1,2] \\
(3-t)(i)+(t-i)[2,3] \\
(4-t)(-i)+(t-3) & [3,4] \end{cases} \]

\[ \gamma : [0,4] \rightarrow \mathbb{C} \]
\[ t \in [0,1] \quad [0,1] \rightarrow AB. \]
\[ \gamma(t) = (1-t) \cdot 1 + t \cdot i \]
\[ t \in [1,2] \quad [1,2] \rightarrow BC. \]
\[ \gamma(t) = (1-(t-1))i + (t-1)(-1) \]
\[ = (2-t)i + (t-1)(-1) \]
\[ t \in [2,3] \quad [2,3] \rightarrow CD. \]
\[ \gamma(t) = (1-(t-2))(-1) + (t-2)(-i) \]
\[ = (3-t)(-1) + (t-2)(-i) \]
\[ t \in [3,4] \quad [3,4] \rightarrow DA \]
\[ \gamma(t) = (1-(t-3))(-i) + (t-3) \cdot 1 \]
\[ = (4-t)(-i) + (t-3) \cdot 1. \]