--- Complex Differentiation

Defn. Let $f: G \to \mathbb{C}$ be a complex function on an open set $G$. The complex derivative of $f$ at $z_0$ is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If $f'(z_0)$ exists, $f(z)$ is differentiable at $z_0$. If $f(z)$ is differentiable in a disk $\{ |z - z_0| < r \}$ at $z_0$, $f(z)$ is holomorphic or analytic at $z_0$. If $f(z)$ is differentiable everywhere in $G$, $f(z)$ is holomorphic or analytic in $G$. If $f(z)$ is holomorphic on $\mathbb{C}$, $f(z)$ is an entire function.

e.g. Show that $f(z) = z^2$ is entire.
\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z
\]

--- Properties of Complex Differentiation

Thm. If \( f(z) \) and \( g(z) \) are differentiable at \( z_0 \), then

- \( f(z) \pm g(z) \) is differentiable at \( z_0 \) and
  \[
  (f \pm g)'(z_0) = f'(z_0) \pm g'(z_0)
  \]

- \( f(z)g(z) \) is differentiable at \( z_0 \) and
  (Product Rule) \( (fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0) \)

Proof. \[
\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0}
\]
\[
\lim_{Z \to Z_0} \frac{f(Z)g(Z) - f(Z)g(Z_0)}{Z - Z_0} + \lim_{Z \to Z_0} \frac{f(Z)g(Z_0) - f(Z_0)g(Z_0)}{Z - Z_0} \\
= \lim_{Z \to Z_0} f(Z) \frac{g(Z) - g(Z_0)}{Z - Z_0} + \lim_{Z \to Z_0} \frac{f(Z) - f(Z_0)}{Z - Z_0} g(Z_0) \\
= f(Z_0)g'(Z_0) + f'(Z_0)g(Z_0)
\]

E.g. Show that \((Z^n)' = nZ^{n-1}\) for all integers \(n\).

Proof. For \(n > 0\), \((1)' = 0\) and \((Z)' = 1\)

Suppose that \((Z^n)' = nZ^{n-1}\)

\((Z^{n+1})' = Z(Z^n)' + (Z)' Z^n = Z(nZ^{n-1}) + Z^n = (n+1)Z^n\)
\[ \lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} + \lim_{z \to z_0} \frac{f(z)g(z_0) - f(z_0)g(z_0)}{z - z_0} \]

\[ = \lim_{z \to z_0} f(z) \frac{g(z) - g(z_0)}{z - z_0} + \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} g(z_0) \]

\[ = f(z_0)g'(z_0) + f'(z_0)g(z_0) \]

e.g. Show that \((z^n)\)' = \(nz^{n-1}\) for all integers \(n\).

Proof. For \(n > 0\), \((1)' = 0\) and \((z)' = 1\).

Suppose that \((z^n)' = nz^{n-1}\).

\[(z^{n+1})' = z(z^n)' + (z)'z^n = z(nz^{n-1}) + z^n = (n+1)z^n\]
For \( n < 0 \),

\[
\left( \frac{1}{z} \right)'' \bigg|_{z_0} = \lim_{z \to z_0} \frac{1}{z} - \frac{1}{z_0} = \lim_{z \to z_0} \frac{Z_0 - Z}{Z - Z_0} = \lim_{z \to z_0} \left( - \frac{1}{z_0} \right) = -\frac{1}{Z_0^2}
\]

\[
\Rightarrow \left( \frac{1}{z} \right)' = -\frac{1}{Z^2}
\]

Suppose that \( \left( \frac{1}{z^n} \right)' = -\frac{n}{z^{n+1}} \)

\[
\frac{1}{z^{n+1}} = \frac{1}{z} \left( \frac{1}{z^n} \right)' + \left( \frac{1}{z} \right)' \left( \frac{1}{z^n} \right)
\]

\[
= \frac{1}{z} \left( -\frac{n}{z^{n+1}} \right) + \left( -\frac{1}{Z^2} \right) \left( \frac{1}{z^n} \right)
\]

\[
= -\frac{n+1}{Z^{n+2}}
\]