1. Elementary Topology of Complex Plane

1.1. **What is topology.** Topology is a branch of geometry that studies the geometric objects, called topological spaces, under continuous maps. Two topological spaces are considered the same if there is a continuous bijection between them. For example, two circles of different radius are regarded as the same topological space.

1.2. **Basic notions of topology.** We mainly concern ourselves with the complex plane $\mathbb{C} \cong \mathbb{R}^2$.

**Definition 1.1.** A set $G \subset \mathbb{C}$ is open if for every point $p \in G$, there is $r > 0$ such that

$$B_p(r) = \{|z - z_0| < r\} \subset G.$$ 

A related concept is the definition of interior points.

**Definition 1.2.** A point $p$ of a set $G \subset \mathbb{C}$ is an interior point of $G$ if there is $r > 0$ such that

$$B_p(r) = \{|z - z_0| < r\} \subset G.$$ 

In other words, $G$ is open if and only if every point of $G$ is an interior point.

**Example 1.3.** Show that $D = \{|z| < 1\}$ is open.

**Proof.** For $z_0 \in D$, let $r = 1 - |z_0|$. For every $z$ satisfying $|z - z_0| < r$,

\begin{equation}
|z| \leq |z - z_0| + |z_0| < r + |z_0| = 1.
\end{equation}

Therefore, $\{|z - z_0| < r\} \subset D$ and hence $D$ is open. \hfill \Box

**Example 1.4.** Show that $D = \{|z| \leq 1\}$ is not open.

**Proof.** For $z_0 = 1 \in D$, $1 + r/2 \notin D$ for all $r > 0$. Therefore,

\begin{equation}
\{|z - 1| < r\} \notin D
\end{equation}

for all $r > 0$ and hence $D$ is not open. \hfill \Box

Here are a couple of facts about open sets:

**Theorem 1.5.** In $\mathbb{C}$,

- $\mathbb{C}$ and $\emptyset$ are open.
- The union of open sets is open.
- The intersection of finitely many open sets is open.
- Let $F : G \to \mathbb{R}$ be a continuous function on an open set $G \subset \mathbb{C}$. Then $\{F(z) < 0\}$ is open.
- Let $F : G \to \mathbb{C}$ be a continuous function on an open set $G \subset \mathbb{C}$. Then $F^{-1}(U)$ is open for every open set $U \subset \mathbb{C}$. 

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Remark 1.6. A subtle point in the above theorem is that the intersection of an infinitely many open sets is not necessarily open. Indeed, for every point \( p \in \mathbb{C} \),
\[
\{p\} = \bigcap_{r > 0} B_p(r) = \bigcap_{r > 0} \{ |z - p| < r \}
\]
That is, every set \( \{p\} \) of a single point is the intersection of open sets; yet \( \{p\} \) is obviously not open. More generally, every closed set \( G \) (see below) is the intersection of open sets given by
\[
G = \bigcap_{r > 0} \left( \bigcup_{p \in G} B_p(r) \right).
\]
The only sets that are both open and closed are \( \mathbb{C} \) and \( \emptyset \) (see below).

Example 1.7. Show that \( A = \{1 < |z| < 2\} \) is open.

Solution. We can write
\[
A = \{1 - |z| < 0\} \cap \{|z| - 2 < 0\} = A_1 \cap A_2.
\]
Since \( F_1(z) = 1 - |z| \) and \( F_2(z) = |z| - 2 \) are continuous functions on \( \mathbb{C} \),
\[
A_1 = \{F_1(z) < 0\} \quad \text{and} \quad A_2 = \{F_2(z) < 0\}
\]
are open by Theorem 1.5. And since the intersection of two open sets is open, \( A \) is open.

Example 1.8. Find \( T^{-1}(D) \) for \( D = |z| < 1/2 \) and
\[
T(z) = \frac{1 + z}{1 - z}.
\]
Show that \( T^{-1}(D) \) is open.

Solution. We have
\[
T^{-1}(D) = \{z : T(z) \in D\} = \left\{ z : |T(z)| < \frac{1}{2} \right\}
\]
\[
= \left\{ z : \left| \frac{1 + z}{1 - z} \right| < \frac{1}{2} \right\} = \{z : 2|1 + z| < |1 - z|\}
\]
\[
= \{ z = x + yi : 4(x + 1)^2 + 4y^2 < (x - 1)^2 + y^2 \}
\]
\[
= \left\{ z = x + yi : (x + \frac{5}{3})^2 + y^2 < \frac{16}{9} \right\}
\]
\[
= \left\{ z : \left| z + \frac{5}{3} \right| < \frac{4}{3} \right\}.
\]
Obviously, \( T^{-1}(D) \) is open.
Definition 1.9. A set $G \subset \mathbb{C}$ is **closed** if its complement $G^c = \mathbb{C}\setminus G$ is open.

A related concept is the definition of **limit points**.

Definition 1.10. A point $p$ is a **limit point** of a set $G \subset \mathbb{C}$ if there exists a sequence $\{p_n\} \subset G$ such that

$$
\lim_{n \to \infty} p_n = p.
$$

In fact, $G$ is closed if and only if $G$ contains all its limit points.

Here are some basic facts about closed sets:

**Theorem 1.11.** In $\mathbb{C}$,

- $\mathbb{C}$ and $\emptyset$ are closed.
- The intersection of closed sets is closed.
- The union of finitely many closed sets is closed.
- Let $F : G \to \mathbb{R}$ be a continuous function on a closed set $G \subset \mathbb{C}$. Then $\{F(z) \leq 0\}$ is closed.
- Let $F : G \to \mathbb{C}$ be a continuous function on a closed set $G \subset \mathbb{C}$. Then $F^{-1}(V)$ is closed for every closed set $V \subset \mathbb{C}$.

Definition 1.12. A set $G \subset \mathbb{C}$ is **bounded** if there is $R > 0$ such that $G \subset \{|z| \leq R\}$; namely, there is $R > 0$ such that $|z| \leq R$ for all $z \in G$. Otherwise, $G$ is **unbounded**.

**Example 1.13.** Show that $G = \{|z - 10| < 10\}$ is bounded.

**Proof.** For every $z \in G$,

$$
|z| \leq |z - 10| + 10 < 10 + 10 = 20.
$$

Therefore, $G$ is bounded. \qed

**Example 1.14.** Show that $G = \{|z - 10| > 10\}$ is unbounded.

**Proof.** There is a sequence $a_n = n + 20$ for $n = 1, 2, \ldots$ such that $a_n \in G$ and

$$
\lim_{n \to \infty} |a_n| = \infty.
$$

Consequently, $G$ is unbounded. \qed

Definition 1.15. A set $G \subset \mathbb{C}$ is **(path-)connected** if for every two points $p, q \in G$, there is a continuous function $\gamma : [0, 1] \to G$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

**Example 1.16.** Show that $D = \{|z| < 1\}$ is connected.
Proof. For every pair of points $p, q \in D$, the line segment
\begin{equation}
\overline{pq} = \{(1 - t)p + tq : 0 \leq t \leq 1\}
\end{equation}
is contained in $D$. Let $\gamma : [0, 1] \to D$ be the function
\begin{equation}
\gamma(t) = (1 - t)p + tq.
\end{equation}
Then $\gamma$ is continuous, $\gamma(0) = p$ and $\gamma(1) = q$. \hfill \Box

Indeed, a set $G$ is called \textit{convex} if $\overline{pq} \subset G$ for all $p, q \in G$; every convex set is connected.

\textbf{Theorem 1.17.} Every convex set in $\mathbb{C}$ is connected.

Proof. Use the same argument as in the above example. \hfill \Box

\textbf{Theorem 1.18.} Let $U$ and $V$ be two connected sets. If $U \cap V \neq \emptyset$, then $U \cup V$ is connected.

Proof. Let $q$ be a point of $U \cap V$. For every pair of points $p_1 \in U$ and $p_2 \in V$, since $U$ and $V$ are connected, there exist continuous functions $\gamma_1 : [0, 1] \to U$ and $\gamma_2 : [0, 1] \to V$ satisfying $\gamma_1(0) = p_1$, $\gamma_1(1) = q$, $\gamma_2(0) = q$ and $\gamma_2(1) = p_2$. Then we define
\begin{equation}
\gamma(t) = \begin{cases} 
\gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\gamma_2(2t - 1) & \text{if } \frac{1}{2} < t \leq 1
\end{cases}
\end{equation}
Clearly, $\gamma : [0, 1] \to U \cup V$ is continuous, $\gamma(0) = p_1$ and $\gamma(1) = p_2$. So $U \cup V$ is connected. \hfill \Box

\textbf{Example 1.19.} Show that $\mathbb{C}\setminus[0, \infty)$ is connected.

Proof. We can write
\begin{equation}
\mathbb{C}\setminus[0, \infty) = U_1 \cup U_2 \cup U_3
\end{equation}
where $U_1 = \{x < 0\}$, $U_2 = \{y > 0\}$ and $U_3 = \{y < 0\}$. All of $U_1, U_2, U_3$ are connected since they are convex.

Since $U_1 \cap U_2 \neq \emptyset$, $U_1 \cup U_2$ is connected. And since $(U_1 \cup U_2) \cap U_3 \neq \emptyset$, $U_1 \cup U_2 \cup U_3$ is connected. \hfill \Box

\textbf{Theorem 1.20.} Let $G$ be a set in $\mathbb{C}$ satisfying $G \subset U \cup V$, where $U$ and $V$ are open, $U \cap V = \emptyset$, $U \cap G \neq \emptyset$ and $V \cap G \neq \emptyset$. Then $G$ is not connected.

\textbf{Example 1.21.} Show that the complement of $\{1 < |z| < 2\}$ in $\mathbb{C}$ is not connected.
Proof. Let \( G = \mathbb{C}\{1 < |z| < 2\} \). Then
\[
G = \{|z| \leq 1\} \cup \{|z| \geq 2\} \subset U \cup V
\]
where \( U = \{|z| < 3/2\} \) and \( V = \{|z| > 3/2\} \). Clearly, \( U \) and \( V \) are open, \( U \cap V = \emptyset \), \( U \cap G \neq \emptyset \) and \( V \cap G \neq \emptyset \). Therefore, \( G \) is not connected.

**Corollary 1.22.** The only sets that are both open and closed in \( \mathbb{C} \) are \( \mathbb{C} \) and \( \emptyset \).

**Proof.** Clearly, \( \mathbb{C} \) is connected since it is convex. Suppose that there is a set \( U \neq \emptyset, \mathbb{C} \) that is both open and closed in \( \mathbb{C} \). Let \( V = U^c = \mathbb{C} \setminus U \). Since \( U \) is closed, \( V \) is open. And since \( U \neq \mathbb{C} \), \( V \neq \emptyset \). Therefore, \( \mathbb{C} \subset U \cup V \), \( U \) and \( V \) are open, \( U \cap V = \emptyset \), \( U \neq \emptyset \) and \( V \neq \emptyset \). It follows from Theorem 1.20 that \( \mathbb{C} \) is not connected. Contradiction. \( \square \)

In complex analysis, a connected open set \( G \) is called a *region* or *domain*. Usually, we study complex functions defined on a region.

Every open set \( G \subset \mathbb{C} \) is a disjoint union of regions:
\[
G = \bigcup_i G_i
\]
where each \( G_i \) is open and connected and \( G_i \cap G_j = \emptyset \) for \( i \neq j \). Each \( G_i \) is a *connected component* of \( G \).