- Continuity

Defn. A complex function \( f(z) \)

is continuous at \( z_0 \) if

\[
\lim_{z \to z_0} f(z) = f(z_0)
\]

Let \( f(z) = u(x, y) + iv(x, y) \)

\[
\lim_{z \to z_0} f(z) = \lim_{(x, y) \to (x_0, y_0)} u(x, y) + iv(x, y)
\]

Thm. \( f(z) \) is continuous at \( z_0 \) if and only if \( u(x, y) \) and \( v(x, y) \) are continuous at \( (x_0, y_0) \).

Cor. \( f(z) \) and \( g(z) \) are continuous

Then \( f(z) \pm g(z), \ f(z)g(z), \ \frac{f(z)}{g(z)} \quad (g(z) \neq 0) \)

and \( f(g(z)) \) are continuous on their domain.
e.g. All polynomials $f(z) = a_0 + a_1 z + \ldots + a_n z^n \in \mathbb{C}[z]$ are continuous everywhere on $\mathbb{C}$.

Rational functions $f(z) = \frac{p(z)}{q(z)} \ (p(z), q(z) \in \mathbb{C}[z])$ are continuous on $\{q(z) \neq 0\}$.

e.g. Show that $|z|$ and $e^z$ are continuous on $\mathbb{C}$.

$|z| = \sqrt{x^2 + y^2}$, $\sqrt{x^2 + y^2}$ continuous on $\mathbb{C}$.

$\Rightarrow |z|$ is continuous on $\mathbb{C}$.

e.g. Show that $e^z = e^x e^{yi} = e^x (\cos y + i \sin y)$;

$e^x \cos y$ and $e^x \sin y$ are continuous.

$\Rightarrow e^z$ continuous on $\mathbb{C}$.

e.g. Show that $e^{f(z)}$ are continuous on $\mathbb{C}$ for every $f(z) \in \mathbb{C}[z]$.

$e^{f(z)} = g(f(z))$ where $g(z) = e^z$. 

$g(z)$ is continuous on $C$?
$f(z)$ is continuous on $C$.

$\Rightarrow$ $g(f(z))$ is continuous on $C$.

E.g. Compute $\lim_{z \to i} (z^3 + z^2 + z)$
Let $f(z) = z^3 + z^2 + z$.

$f(z)$ is continuous at $i$.

$\Rightarrow$ $\lim_{z \to i} f(z) = f(i) = -1$.

E.g. Let $f(z) = \begin{cases} \frac{z^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Show that $f(z)$ is not continuous at $0$.

E.g. Let $f(z) = \begin{cases} \frac{z^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Show that $f(z)$ is continuous on $C$. 
e.g. Let \( f(z) = \begin{cases} 1 & \text{if } x, y \in \mathbb{Q} \\ 0 & \text{if one of } x, y \notin \mathbb{Q} \end{cases} \)

Show that \( f(z) \) is nowhere continuous.

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**Logarithmic Functions**

\[
e^z = e^x e^{yi} = e^x (\cos y + i \sin y)
\]

\[
e^x = |e^z|, \quad y = \arg(e^z)
\]

Let \( w = e^z \).

\[
z = \log w = x + yi
\]

\[
= \ln |w| + i \arg(w)
\]

**Defn.** \( \log z = \ln |z| + i \arg(z) \)

is the complex natural log function on \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). \( \log z \) is multi-valued.
\[ \log z = \{ \ln r + i\theta + 2m\pi i : m \in \mathbb{Z} \} \]
for \( z = re^{i\theta} \)

**Property.** \( e^{\log z} = z \)

\[ \log (e^z) = z + 2m\pi i \]

**e.g.** \( \log (1) = 2m\pi i \)

\[ \log (-1) = \pi i + 2m\pi i \]

\[ \log (-100) = \ln(100) + \pi i + 2m\pi i \]

**e.g.** Solve \( z^3 = 1 \)

\[ \log (z^3) = \log 1 = 2m\pi i \]

\[ 3 \log z = 2m\pi i \quad \Rightarrow \quad \log z = \frac{2m\pi i}{3} \]

\[ \log z + \log z + \log z \]
\[ z = \left( \frac{2\pi i}{3} \right) \]

\[ \sqrt[n]{z} = e^{\frac{1}{n} \log z} \]

--- Principal Logarithm

\[ \log z = \ln |z| + i \arg z \]

\[-\pi < \arg z \leq \pi\]

is the principal logarithm or the principal branch of \( \log z \)

e.g. \( \log (-1) = \pi i \)

\( \log (-\i) = -\frac{\pi}{2} \i \)

Caution. \( \log (z_1z_2) = \log z_1 + \log z_2 \)

\[ z \to e^z \]

\[ \downarrow \]

\[ z \to c \]