Solutions for Math 311 Assignment #9

(1) Compute the integrals of the following functions along the curves $C_1 = \{|z| = 1\}$ and $C_2 = \{|z - 2| = 1\}$, both oriented counterclockwise:

(a) $\frac{1}{2z - z^2}$

(b) $\frac{\sinh z}{(2z - z^2)^2}$

Solution.

(a) 
\[
\int_{|z|=1} \frac{dz}{2z - z^2} = \int_{|z|=1} \frac{(2 - z)^{-1}}{z} \, dz = 2\pi i (2 - 0)^{-1} = \pi i
\]

(b) 
\[
\int_{|z|=1} \frac{\sinh z}{(2z - z^2)^2} \, dz = \int_{|z|=1} \frac{(\sinh z)(2 - z)^{-2}}{z^2} \, dz
\]
\[
= 2\pi i ((\sinh z)(2 - z)^{-2})' \bigg|_{z=0} = \frac{\pi i}{2}
\]

(2) Show that if $f$ is analytic inside and on a simple closed curve $C$ and $z_0$ is not on $C$, then

\[(n - 1)! \int_C \frac{f^{(m)}(z)}{(z - z_0)^n} \, dz = (m + n - 1)! \int_C \frac{f(z)}{(z - z_0)^{m+n}} \, dz\]

for all positive integers $m$ and $n$.

Proof. If $z_0$ lies outside $C$, then

\[
\int_C \frac{f^{(m)}(z)}{(z - z_0)^n} \, dz = \int_C \frac{f(z)}{(z - z_0)^{m+n}} \, dz = 0
\]

by Cauchy Integral Theorem, since $f^{(m)}z/(z-z_0)^n$ and $f(z)/(z-z_0)^{m+n}$ are analytic on and inside $C$.

If $z_0$ lies inside $C$, then

\[(n - 1)! \int_C \frac{f^{(m)}(z)}{(z - z_0)^n} \, dz = (f^{(m)}(z))^{(n-1)} \bigg|_{z=z_0} = f^{(m+n-1)}(z_0)
\]

and

\[(m + n - 1)! \int_C \frac{f(z)}{(z - z_0)^{m+n}} \, dz = f^{(m+n-1)}(z_0)\]
by Cauchy Integral Formula. Therefore,

\[(n - 1)! \int_C \frac{f^{(m)}(z)}{(z - z_0)^n} \, dz = (m + n - 1)! \int_C \frac{f(z)}{(z - z_0)^{m+n}} \, dz.\]

\[\square\]

(3) Let \(f(z)\) be an entire function. Show that \(f(z)\) is a constant if \(|f(z)| \leq \ln(|z| + 1)\) for all \(z \in \mathbb{C}\).

**Proof.** For every \(z_0 \in \mathbb{C}\), we have

\[f'(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2} \, dz\]

for all \(R > 0\). Since

\[\left| \frac{f(z)}{(z-z_0)^2} \right| \leq \frac{\ln(|z| + 1)}{R^2} \leq \frac{\ln(R + |z_0|) + 1}{R^2}\]

for \(|z - z_0| = R\),

\[|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2} \, dz \right| \leq \frac{\ln(R + |z_0|) + 1}{R}.\]

And since

\[\lim_{R \to \infty} \frac{\ln(R + |z_0|) + 1}{R} = \lim_{R \to \infty} \frac{1}{R + |z_0| + 1} = 0,\]

by L’Hospital, we conclude that \(|f'(z_0)| = 0\) and hence \(f'(z_0) = 0\) for every \(z_0 \in \mathbb{C}\). Therefore, \(f(z)\) is a constant. \(\square\)

(4) Compute the integral

\[\int_0^\infty \frac{x \, dx}{x^3 + 1}.\]

**Solution.** Consider the contour integral of \(z/(z^3 + 1)\) along \(L_R = [0, R], C_R = \{z = Re^{it} : 0 \leq t \leq 2\pi/3\}\) and \(M_R = \{te^{2\pi i/3} : 0 \leq t \leq R\}\). By CIT,

\[\int_{L_R} \frac{z \, dz}{z^3 + 1} + \int_{C_R} \frac{z \, dz}{z^3 + 1} - \int_{M_R} \frac{z \, dz}{z^3 + 1} = \int_{|z-e^{2\pi i/3}|=1/2} \frac{z \, dz}{z^3 + 1}\]
By CIF,
\[ \int_{|z-e^{\pi i/3}|=1/2} \frac{zdz}{z^3+1} = \frac{2\pi i \exp(\pi i/3)}{(\exp(\pi i/3) + 1)(\exp(\pi i/3) - \exp(-\pi i/3))} = \frac{2\pi \exp(\pi i/3)}{(\exp(\pi i/3) + 1)\sqrt{3}}. \]

For \( z \) lying on \( C_R \),
\[ \left| \frac{z}{z^3 + 1} \right| \leq \frac{R}{R^3 - 1} \]
and hence
\[ \left| \int_{C_R} \frac{zdz}{z^3 + 1} \right| \leq \frac{2\pi R}{3(R^3 - 1)} \]

It follows that
\[ \lim_{R \to \infty} \int_{C_R} \frac{zdz}{z^3 + 1} = 0 \]

And
\[ \int_{M_R} \frac{zdz}{z^3 + 1} = \exp(4\pi i/3) \int_0^R \frac{xdx}{x^3 + 1} \]

Therefore, we have
\[ (1 - \exp(4\pi i/3)) \int_0^\infty \frac{xdx}{x^3 + 1} = \frac{2\pi \exp(\pi i/3)}{(\exp(\pi i/3) + 1)\sqrt{3}} \]
and hence
\[ \int_0^\infty \frac{xdx}{x^3 + 1} = \frac{2\pi}{3\sqrt{3}}. \]

(5) Compute the integral
\[ \int_0^\infty \frac{\cos x}{x^4 + 1} dx. \]

**Solution.** Since \( \cos x / (x^4 + 1) \) is even,
\[ \int_0^\infty \frac{\cos x}{x^4 + 1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos x}{x^4 + 1} dx. \]

Actually, we have
\[ \int_0^\infty \frac{\cos x}{x^4 + 1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{ix}}{x^4 + 1} dx \]
since \( e^{ix} = \cos x + i \sin x \).
Consider the contour integral of \(e^{iz}/(z^4 + 1)\) along the path 
\(L_R = [-R, R]\) and \(C_R = \{|z| = R, \text{Im}(z) \geq 0\}\), oriented counterclockwise. By CIT, we have

\[
\int_{L_R} \frac{e^{iz}}{z^4 + 1} \, dz + \int_{C_R} \frac{e^{iz}}{z^4 + 1} \, dz = \int_{|z - e^{\pi i/4}| = 1/2} \frac{e^{iz}}{z^4 + 1} \, dz + \int_{|z - e^{3\pi i/4}| = 1/2} \frac{e^{iz}}{z^4 + 1} \, dz
\]

By CIF,

\[
\int_{|z - e^{\pi i/4}| = 1/2} \frac{e^{iz}}{z^4 + 1} \, dz = \frac{2\pi ie^{i(\sqrt{2}+i\sqrt{2})/2}}{(e^{\pi i/4} - e^{3\pi i/4})(e^{\pi i/4} - e^{-\pi i/4})} = \frac{\pi (1 - i) \exp((-\sqrt{2} + i\sqrt{2})/2)}{2\sqrt{2}}
\]

and similarly,

\[
\int_{|z - e^{3\pi i/4}| = 1/2} \frac{e^{iz}}{z^4 + 1} \, dz = \frac{\pi (1 + i) \exp((-\sqrt{2} - i\sqrt{2})/2)}{2\sqrt{2}}
\]

Therefore,

\[
\int_{L_R} \frac{e^{iz}}{z^4 + 1} \, dz + \int_{C_R} \frac{e^{iz}}{z^4 + 1} \, dz = \frac{\pi e^{-\sqrt{2}/2}}{\sqrt{2}} (\cos(\sqrt{2}/2) + \sin(\sqrt{2}/2)).
\]

For \(z\) lying on \(C_R\), \(y = \text{Im}(z) \geq 0\) and hence \(|e^{iz}| = e^{-y} \leq 1\). Hence

\[
\left| \frac{e^{iz}}{z^4 + 1} \right| \leq \frac{1}{R^4 - 1}
\]

and it follows that

\[
\left| \int_{C_R} \frac{e^{iz}}{z^4 + 1} \, dz \right| \leq \frac{\pi R}{R^4 - 1}
\]

Since

\[
\lim_{R \to \infty} \frac{\pi R}{R^4 - 1} = 0,
\]

we conclude that

\[
\lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{z^4 + 1} \, dz = 0.
\]
Therefore,
\[
\int_0^\infty \frac{\cos x}{x^4 + 1} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^4 + 1} \, dx
\]
\[
= \frac{1}{2} \lim_{R \to \infty} \int_{LR} \frac{e^{iz}}{z^4 + 1} \, dz
\]
\[
= \frac{\pi e^{-\sqrt{2}/2}}{2\sqrt{2}} (\cos(\sqrt{2}/2) + \sin(\sqrt{2}/2)).
\]

(6) Compute the integral
\[
\int_{-\pi}^{\pi} \frac{dx}{2 - (\cos x + \sin x)}.
\]

Solution. Let \( z = e^{ix} \). Then \( dz = ie^{ix} \, dx, \, dx = -idz/z \) and hence
\[
\int_{-\pi}^{\pi} \frac{dx}{2 - (\cos x + \sin x)} = \int_{|z|=1} \frac{dz}{2 - (e^{ix} + e^{-ix})/2 - (e^{ix} - e^{-ix})/(2i)}
\]
\[
= \int_{|z|=1} \frac{dz}{2z - (z^2 + 1)/2 - (z^2 - 1)/(2i)}
\]
\[
= (i - 1) \int_{|z|=1} \frac{dz}{z^2 - 2(1 + i)z + i}
\]
\[
= (i - 1) \int_{|z|=1} \frac{dz}{(z - z_1)(z - z_2)}
\]
\[
= \frac{2\pi i(i - 1)}{z_2 - z_1} = \sqrt{2}\pi
\]
where \( z_1 = (1 + \sqrt{2}/2) + (1 + \sqrt{2}/2)i \) and \( z_2 = (1 - \sqrt{2}/2) + (1 - \sqrt{2}/2)i \).

(7) Let \( f(z) \) be a complex polynomial of degree at least 2 and \( R \) be a positive number such that \( f(z) \neq 0 \) for all \( |z| \geq R \). Show that
\[
\int_{|z|=R} \frac{dz}{f(z)} = 0.
\]

Proof. Let \( f(z) = a_0 + a_1z + ... + a_nz^n \), where \( a_n \neq 0 \) and \( n = \deg f \). Since \( f(z) \neq 0 \) for \( |z| \geq R \), \( 1/f(z) \) is analytic in \( |z| \geq R \). Hence
\[
\int_{|z|=R} \frac{dz}{f(z)} = \int_{|z|=r} \frac{dz}{f(z)}
\]
for all $r \geq R$ by Cauchy Integral Theorem. Since

$$|f(z)| \geq |a_n| |z|^n - |a_{n-1}| |z|^{n-1} - ... - |a_0|$$

we have

$$\left| \frac{1}{f(z)} \right| \leq \frac{1}{|a_n| r^n - |a_{n-1}| r^{n-1} - ... - |a_0|}$$

for $|z| = r$ sufficiently large. It follows that

$$\left| \int_{|z|=r} \frac{dz}{f(z)} \right| \leq \frac{2\pi r}{|a_n| r^n - |a_{n-1}| r^{n-1} - ... - |a_0|}$$

And since $n \geq 2$,

$$\lim_{r \to \infty} \frac{2\pi r}{|a_n| r^n - |a_{n-1}| r^{n-1} - ... - |a_0|} = \lim_{r \to \infty} \frac{2\pi}{n |a_n| r^{n-1} - (n-1) |a_{n-1}| r^{n-2} - ... - |a_1|} = 0$$

by L'Hospital. Hence

$$\int_{|z|=R} \frac{dz}{f(z)} = \lim_{r \to \infty} \int_{|z|=r} \frac{dz}{f(z)} = 0.$$

(8) Let $f(z)$ be an entire function satisfying

$$|f(z_1 + z_2)| \leq |f(z_1)| + |f(z_2)|$$

for all complex numbers $z_1$ and $z_2$. Show that $f(z)$ is a polynomial of degree at most 1.
Proof. We have
\[ \sum_{k=1}^{n} f(z_k) = f(z_1) + f(z_2) + \sum_{k=3}^{n} f(z_k) \]
\[ = f(z_1 + z_2) + \sum_{k=3}^{n} f(z_k) \]
\[ = f(z_1 + z_2) + f(z_3) + \sum_{k=4}^{n} f(z_k) \]
\[ = f(z_1 + z_2 + z_3) + \sum_{k=4}^{n} f(z_k) \]
\[ = \ldots = f(z_1 + z_2 + \ldots + z_n) = f\left( \sum_{k=1}^{n} x_k \right). \]

Therefore,
\[ \sum_{k=1}^{n} f(z_k) = f(z_1) + f(z_2) + \ldots + f(z_n) = f(z_1 + z_2 + \ldots + z_n) = f\left( \sum_{k=1}^{n} x_k \right) \]
for all complex numbers \( z_1, z_2, \ldots, z_n \). Particularly, this holds for \( z_1 = z_2 = \ldots = z_n = z/n \):
\[ nf\left( \frac{z}{n} \right) = f(z) \]
for all \( z \in \mathbb{C} \) and all positive integer \( n \). Let \( M \) be the maximum of \( |f(z)| \) for \( |z| = 1 \). Then
\[ |f(z)| = n \left| f\left( \frac{z}{n} \right) \right| \leq nM \]
for all \( z \) satisfying \( |z| = n \).

By CIF,
\[ f''(z_0) = \frac{1}{\pi i} \int_{|z|=n} \frac{f(z)}{(z-z_0)^3} dz \]
for \( |z_0| < n \). Since
\[ \left| \frac{f(z)}{(z-z_0)^3} \right| = \frac{|f(z)|}{|z-z_0|^3} \leq \frac{nM}{(n-|z_0|)^3} \]
for \( |z| = n \) and \( |z_0| < n \),
\[ \left| \frac{1}{\pi i} \int_{|z|=n} \frac{f(z)}{(z-z_0)^3} dz \right| \leq \frac{2n^2M}{(n-|z_0|)^3}. \]
And since
\[
\lim_{n \to \infty} \frac{2n^2M}{(n - |z_0|)^3} = \lim_{n \to \infty} \frac{2M/n}{(1 - |z_0|/n)^3} = 0,
\]
we conclude that \( f''(z_0) = 0 \) for all \( z_0 \). Therefore, \( f'(z) \equiv a \) is a constant and \( f(z) = az + b \) is a polynomial of degree at most 1. □