Solutions for Math 311 Assignment #8

(1) Let $C$ be the boundary of the triangle with vertices at the points $0, 3i$ and $-4$ oriented counterclockwise. Compute the contour integral
\[ \int_C (e^z - z) dz. \]

Solution. By Cauchy Integral Theorem, $\int_C e^z dz = 0$ since $C$ is closed and $e^z$ is entire. Therefore,
\[
\int_C (e^z - z) dz = -\int_C \overline{z} dz = -\int_{p_1}^{p_2} \overline{z} dz - \int_{p_2}^{p_3} \overline{z} dz - \int_{p_3}^{p_1} \overline{z} dz
\]
\[
= -\int_0^1 (-3it)(3it) - \int_0^1 (-3i(1 - t) - 4t)(3i(1 - t) - 4t)
\]
\[
- \int_0^1 (-4)(1 - t)d((-4)(1 - t))
\]
\[
= -\frac{9}{2} - \frac{7}{2} - 12i + 8 = -12i
\]
where $p_1 = 0, p_2 = 3i$ and $p_3 = -4$.

(2) Compute
\[ \int_{-1}^{1} z^i dz \]
where the integrand denote the principal branch
\[ z^i = \exp(i \text{Log } z) \]
of $z^i$ and where the path of integration is any continuous curve from $z = -1$ to $z = 1$ that, except for its starting and ending points, lies below the real axis.

Solution. Note that $z^{i+1}/(i + 1)$ is an anti-derivative of $z^i$ outside the branch locus $(-\infty, 0]$. So
\[
\int_{-1}^{1} z^i dz = \frac{z^{i+1}}{i+1} \bigg|_{-1}^{1} - \lim_{\ln(z) \to 0}^{z^{i+1}} \frac{z^{i+1}}{i+1}
\]
\[
= \frac{1}{i + 1} - \frac{\exp((i + 1)(-\pi i))}{i + 1}
\]
\[
= \frac{1 + e^\pi}{i + 1} = \frac{1 + e^\pi}{2}(1 - i)
\]
(3) Apply Cauchy Integral Theorem to show that
\[ \int_C f(z) \, dz = 0 \]
when \( C \) is the unit circle \(|z| = 1\), in either direction, and when
(a) \( f(z) = \frac{z^3}{z^2 + 5z + 6} \);
(b) \( f(z) = e^{\tan z} \);
(c) \( f(z) = \log(z + 3i) \).

**Solution.** By Cauchy Integral Theorem, \( \int_{|z|=1} f(z) \, dz = 0 \) if
\( f(z) \) is analytic on and inside the circle \(|z| = 1\). Hence it is
enough to show that \( f(z) \) is analytic in \( \{|z| \leq 1\} \).

(a) \( f(z) \) is analytic in \( \{z \neq -2, -3\} \) and hence analytic in
\( \{|z| \leq 1\} \).

(b) \( f(z) \) is analytic in \( \{z : \cos z = 0\} = \{z = n\pi + \pi/2, n \in \mathbb{Z}\} \).
Since \(|n\pi + \pi/2| > 1\) for all integers \( n \), \( f(z) \) is analytic in
\( \{|z| \leq 1\} \).

(c) \( \log(z) \) is analytic in \( \mathbb{C}\setminus(-\infty, 0] \) and hence \( \log(z + 3i) \) is
analytic in \( \mathbb{C}\setminus\{z : z = x - 3i, x \in (-\infty, 0]\} \). Since \(|x-3i| > 1\)
for all \( x \) real, \( f(z) \) is analytic in \( \{|z| \leq 1\} \).

(4) Let \( C_1 \) denote the positively oriented boundary of the curve
given by \(|x| + |y| = 2\) and \( C_2 \) be the positively oriented circle
\(|z| = 4\). Apply Cauchy Integral Theorem to show that
\[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \]
when
(a) \( f(z) = \frac{z + 1}{z^2 + 1} \);
(b) \( f(z) = \frac{z + 2}{\sin(z/2)} \);
(c) \( f(z) = \frac{\sin(z)}{z^2 + 6z + 5} \).

**Solution.** By Cauchy Integral Theorem, \( \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \)
if \( f(z) \) is analytic on and between \( C_1 \) and \( C_2 \). Hence it is enough
to show that \( f(z) \) is analytic in \(|x| + |y| \geq 2, |z| \leq 4\).

(a) \( f(z) \) is analytic in \( \{z \neq \pm i\} \). Since \( \pm i \in \{|x| + |y| < 2\} \),
\( f(z) \) is analytic in \( \{|x| + |y| \geq 2, |z| \leq 4\} \).

(b) \( f(z) \) is analytic in \( \{z : \sin(z/2) \neq 0\} = \{z \neq 2n\pi : n \in \mathbb{Z}\} \).
Since \( 2n\pi \in \{|x| + |y| < 2\} \) for \( n = 0 \) and \(|2n\pi| > 4\) for
\( n \neq 0 \) and \( n \in \mathbb{Z} \), \( f(z) \) is analytic in \(|x| + |y| \geq 2, |z| \leq 4\}. \)
(c) $f(z)$ is analytic in $\{ z \neq -1, -5 \}$. Since $-1 \in \{ |x| + |y| < 2 \}$ for $n = 0$ and $|-5| > 4$, $f(z)$ is analytic in $\{ |x| + |y| \geq 2, |z| \leq 4 \}$.

(5) Let $C$ denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals

(a) $\int_C \frac{zdz}{z + 1}$;

(b) $\int_C \frac{\cosh z}{z^2 + z}dz$;

(c) $\int_C \frac{\tan(z/2)}{z - \pi/2}dz$.

Solution.

(a) By CIF,

$$\int_C \frac{zdz}{z + 1} = 2\pi i(-1) = -2\pi i.$$

(b) By CIT,

$$\int_C \frac{\cosh z}{z^2 + z}dz = \int_{|z|=r} \frac{\cosh z}{z^2 + z}dz + \int_{|z+1|=r} \frac{\cosh z}{z^2 + z}dz$$

for $r = 1/2$. By CIF,

$$\int_{|z|=r} \frac{\cosh z}{z^2 + z}dz = 2\pi i \left. \frac{\cosh(z)}{z + 1} \right|_{z=0} = 2\pi i$$

and

$$\int_{|z+1|=r} \frac{\cosh z}{z^2 + z}dz = 2\pi i \left. \frac{\cosh z}{z} \right|_{z=-1} = -2\pi i \cosh(-1).$$

Hence

$$\int_C \frac{\cosh z}{z^2 + z}dz = 2\pi i(1 - \cosh(-1)).$$

(c) Note that $\tan(z/2)$ is analytic in $\{ z \neq (2n + 1)\pi : n \in \mathbb{Z} \}$ and hence analytic inside $C$. Therefore,

$$\int_C \frac{\tan(z/2)}{z - \pi/2}dz = 2\pi i \tan(\pi/4) = 2\pi i$$

by CIF.

(6) Find the value of the integral $g(z)$ around the circle $|z - i| = 2$ oriented counterclockwise when
(a) \( g(z) = \frac{1}{z^2 + 4} \);

(b) \( g(z) = \frac{1}{z(z^2 + 4)} \).

**Solution.**

(a) Since \(-2i \not\in \{|z - i| \leq 2\} \) and \(2i \in \{|z - i| \leq 2\} \),

\[
\int_{|z-i|=2} g(z)dz = \int_{|z-i|=2} \frac{(z + 2i)^{-1}}{z - 2i}dz = 2\pi i (2i + 2i)^{-1} = \frac{\pi}{2}
\]

by CIF.

(b) By CIT,

\[
\int_{|z-i|=2} g(z)dz = \int_{|z|=r} g(z)dz + \int_{|z-2i|=r} g(z)dz
\]

for \(r < 1/2\). Since

\[
\int_{|z|=r} g(z)dz = 2\pi i \left. \frac{1}{z^2 + 4} \right|_{z=0} = \frac{\pi i}{2}
\]

and

\[
\int_{|z-2i|=r} g(z)dz = 2\pi i \left. \frac{1}{z(z + 2i)} \right|_{z=2i} = -\frac{\pi i}{4}
\]

by CIF,

\[
\int_{|z-i|=2} g(z)dz = \frac{\pi i}{4}
\]