Solutions for Math 311 Assignment #5

(1) Explain why the function \( f(z) = 2z^2 - 3 - ze^z + e^{-z} \) is entire.

\textit{Proof.} Since every polynomial is entire, \( 2z^2 - 3 \) is entire; since both \(-z\) and \(e^z\) are entire, their product \(-ze^z\) is entire; since \(e^z\) and \(-z\) are entire, their composition \(e^{-z}\) is entire. Finally, \( f(z) \) is the sum of \(2z^2 - 3, -ze^z\) and \(e^{-z}\) and hence entire. \(\Box\)

(2) Let \( f(z) \) be an analytic function on a connected open set \( D \). If there are two constants \( c_1 \) and \( c_2 \in \mathbb{C} \), not all zero, such that \( c_1 f(z) + c_2 \overline{f(z)} = 0 \) for all \( z \in D \), then \( f(z) \) is a constant on \( D \).

\textit{Proof.} If \( c_2 = 0 \), \( c_1 \neq 0 \) since \( c_1 \) and \( c_2 \) cannot be both zero. Then we have \( c_1 f(z) = 0 \) and hence \( f(z) = 0 \) for all \( z \in D \).

If \( c_2 \neq 0 \), \( \overline{f(z)} = -(c_1/c_2) f(z) \). And since \( f(z) \) is analytic in \( D \), \( \overline{f(z)} \) is analytic in \( D \). So both \( f(z) \) and \( \overline{f(z)} \) are analytic in \( D \). Therefore, both \( f(z) \) and \( \overline{f(z)} \) satisfy Cauchy-Riemann equations in \( D \). Hence

\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u + vi) = 0
\]

and

\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u - vi) = 0
\]

in \( D \), where \( f(z) = u(x,y) + iv(x,y) \) with \( u = \text{Re}(f) \) and \( v = \text{Im}(f) \). It follows that

\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) v = 0
\]

and hence \( u_x = u_y = v_x = v_y = 0 \) in \( D \). Therefore, \( u \) and \( v \) are constants on \( D \) and hence \( f(z) \equiv \text{const.} \) \(\Box\)

(3) Show that the function \( \sin(z) \) is nowhere analytic on \( \mathbb{C} \).

\textit{Proof.} Since

\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \sin(z) = \frac{\partial}{\partial x} \sin(z) + i \frac{\partial}{\partial y} \sin(z)
\]

\[
= \cos(z) \frac{\partial z}{\partial x} + i \cos(z) \frac{\partial z}{\partial y}
\]

\[
= \cos(z) + i \cos(z)(-i) = 2 \cos(z)
\]

\( \sin(z) \) is not differentiable and hence not analytic at every point \( z \) satisfying \( \cos(z) \neq 0 \). At every point \( z_0 \) satisfying \( \cos(z_0) = 0 \), i.e., \( z_0 = n\pi + \pi/2 \), \( \sin(z) \) is not differentiable in \( |z - z_0| < r \) for all \( r > 0 \). Hence \( \sin(z) \) is not analytic at \( z_0 = n\pi + \pi/2 \) either. In conclusion, \( \sin(z) \) is nowhere analytic. \( \square \)

(4) Show that
\[
| \exp(z^3 + i) + \exp(-iz^2) | \leq e^{x^3 - 3xy^2} + e^{2xy}
\]
where \( x = \text{Re}(z) \) and \( y = \text{Im}(z) \).

Proof. Note that \( |e^z| = e^{\text{Re}(z)} \). Therefore,
\[
| \exp(z^3 + i) + \exp(-iz^2) |
\leq | \exp(z^3 + i) | + | \exp(-iz^2) |
= \exp(\text{Re}(z^3 + i)) + \exp(\text{Re}(-iz^2))
= \exp(\text{Re}(x^3 - 3xy^2 + (3x^2y - y^3 + 1)i))
+ \exp(\text{Re}(2xy - (x^2 - y^2)i))
= e^{x^3 - 3xy^2} + e^{2xy}.
\] \( \square \)

(5) Show that
\[
\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - (\tan z_1)(\tan z_2)}
\]
for all complex numbers \( z_1 \) and \( z_2 \) satisfying \( z_1, z_2, z_1 + z_2 \neq n\pi + \pi/2 \) for any integer \( n \).

Proof. Since
\[
\tan z_1 + \tan z_2 = \frac{i(e^{-iz_1} - e^{iz_1})}{e^{iz_1} + e^{-iz_1}} + \frac{i(e^{-iz_2} - e^{iz_2})}{e^{iz_2} + e^{-iz_2}}
= i \frac{(e^{-iz_1} - e^{iz_1})(e^{iz_2} + e^{-iz_2}) + (e^{-iz_2} - e^{iz_2})(e^{iz_1} + e^{-iz_1})}{(e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2})}
= -2i \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{(e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2})}
\]
and
\[
1 - (\tan z_1)(\tan z_2) = 1 - \left(\frac{i(e^{-iz_1} - e^{iz_1})}{e^{iz_1} + e^{-iz_1}}\right)\left(\frac{i(e^{-iz_2} - e^{iz_2})}{e^{iz_2} + e^{-iz_2}}\right)
\]
\[
= \frac{(e^{-iz_1} + e^{iz_1})(e^{-iz_2} + e^{iz_2}) + (e^{-iz_1} - e^{-iz_1})(e^{-iz_2} - e^{iz_2})}{(e^{-iz_1} + e^{iz_1})(e^{iz_2} + e^{-iz_2})}
\]
\[
= 2\frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{(e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2})},
\]
we have
\[
\tan z_1 + \tan z_2 = \frac{-e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}} = \tan(z_1 + z_2).
\]

Alternatively, we can argue as follows if we assume that the identity holds for \(z_1\) and \(z_2\) real. Let
\[
F(z_1, z_2) = \tan(z_1 + z_2) - \frac{\tan z_1 + \tan z_2}{1 - (\tan z_1)(\tan z_2)}.
\]

We assume that \(F(z_1, z_2) = 0\) for all \(z_1, z_2 \in \mathbb{R}\) with \(z_1, z_2, z_1 + z_2 \neq n\pi + \pi/2\).

Fixing \(z_1 \in \mathbb{R}\), we let \(f(z) = F(z_1, z)\). Then \(f(z)\) is analytic in its domain \(\mathbb{C}\setminus\{n\pi + \pi/2\} \cup \{n\pi + \pi/2 - z_1\}\). And we know that \(f(z) = 0\) for \(z\) real. Therefore, by the uniqueness of analytic functions, \(f(z) \equiv 0\) in its domain. So \(F(z_1, z_2) = 0\) for all \(z_1 \in \mathbb{R}\) and \(z_2 \in \mathbb{C}\) in its domain.

Fixing \(z_2 \in \mathbb{C}\), we let \(g(z) = F(z, z_2)\). Then \(g(z)\) is analytic in its domain \(\mathbb{C}\setminus\{n\pi + \pi/2\} \cup \{n\pi + \pi/2 - z_2\}\). And we have proved that \(g(z) = 0\) for \(z\) real. Therefore, by the uniqueness of analytic functions, \(g(z) \equiv 0\) in its domain. So \(F(z_1, z_2) = 0\) for all \(z_1 \in \mathbb{C}\) and \(z_2 \in \mathbb{C}\) in its domain. \(\square\)

(6) Show that the entire function \(\cosh(z)\) takes every value in \(\mathbb{C}\) infinitely many times.

**Proof.** For every \(w_0 \in \mathbb{C}\), the quadratic equation \(y^2 - 2w_0y + 1 = 0\) has a complex root \(y_0\). We cannot have \(y_0 = 0\) since \(0^2 - 2w_0 \cdot 0 + 1 \neq 0\). Therefore, \(y_0 \neq 0\) and there is \(z_0 \in \mathbb{C}\) such that \(e^{iz_0} = y_0\). Then
\[
\cosh(z_0) = \frac{e^{iz_0} + e^{-iz_0}}{2} = \frac{y_0^2 + 1}{2y_0} = \frac{2w_0y_0}{2y_0} = w_0.
\]
And since \( \cosh(z + 2\pi i) = \cosh(z) \), \( \cosh(z_0 + 2n\pi i) = w_0 \) for all integers \( n \). Therefore, \( \cosh(z) \) takes every value \( w_0 \) infinitely many times. \( \square \)