Math 311 Final Review

Some information on the final:

- Time and location: 9:00-12:00, Dec. 15, 2011, MEC 23
- Sections covered: (BC) Sec. 1-25, 27, 29-53, 55-70, 72-76 (BMPS)
  1.0-1.5, 2.1-2.4, 3.4-3.5, 4.1-4.3, 5.1-5.3, 7.1-7.4, 8.1-8.3, 9.1-9.2

(1) Let $f(z)$ be the principal branch of $\sqrt[3]{z}$.
   (a) Find $f(-i)$.
   (b) Show that 
       $$f(z_1)f(z_2) = \lambda f(z_1z_2)$$
       for all $z_1, z_2 \neq 0$, where $\lambda = 1, \frac{-1 + \sqrt{3}i}{2}$ or $\frac{-1 - \sqrt{3}i}{2}$.

(2) Let $f(z)$ be an entire function satisfying that $|f(z)| \leq |z|^2$ for all $z$. Show that $f(z) \equiv az^2$ for some constant $a$ satisfying $|a| \leq 1$.

(3) Let 
    $$f(z) = \frac{z^2}{z^2 - z - 2}$$

    Find the Laurent series of $f(z)$ in each of the following domains:
    (a) $1 < |z| < 2$
    (b) $0 < |z - 2| < 1$

(4) Do the following:
   (a) Find $\cos\left(\frac{\pi}{3} + i\right)$.
   (b) Find the Taylor series of $(\cos z)^2$ at $z = \pi$.
   (c) Show that 
       $$|\cos(z)|^2 = (\cos x)^2 + (\sinh y)^2$$
       for all $z \in \mathbb{C}$, where $x = \text{Re}(z)$ and $y = \text{Im}(z)$.
   (d) Let $C_N$ be the boundary of the square 
       $$\{|x| \leq N\pi, |y| \leq N\pi\},$$
       where $N$ is a positive integer. Show that 
       $$\lim_{N \to \infty} \int_{C_N} \frac{dz}{z^3 \cos z} = 0$$
(5) Let $C$ be the circle $|z| = 1$ oriented counter-clockwise.

(a) Compute
\[
\int_C \frac{1}{z^2 - 8z + 1} \, dz
\]

(b) Use or not use part (a) to compute
\[
\int_0^\pi \frac{1}{4 - \cos \theta} \, d\theta
\]

(6) For each of the following complex functions, do the following:
- find all its singularities in $\mathbb{C}$;
- write the principal part of the function at each singularity;
- for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
- compute the residue of the function at each singularity.

(a) $f(z) = (1 - z^2) \exp \left( \frac{1}{z} \right)$

(b) $f(z) = \frac{1}{(\sin z)^2}$

(c) $f(z) = \frac{1 - \cos z}{z^2}$

(d) $f(z) = \frac{e^z}{z(z - 1)^2}$

(7) Compute the following contour integrals.

(a) $\int_L zdz$, where $L$ is the boundary of the triangle $ABC$ with $A = 0$, $B = 1$ and $C = i$, oriented counter-clockwise.

(b) $\int_C z^{2008} \frac{1}{z^{2009} + z + 1} \, dz$, where $C$ is the circle $|z| = 2$ oriented counter-clockwise.

(8) Let $f(z) = u(x, y) + iv(x, y)$ be an entire function satisfying $u(x, y) \leq x$ for all $z = x + yi$. Show that $f(z)$ is a polynomial of degree at most one.

(9) Compute the integral $\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1} \, dx$