Solutions for Math 311 Assignment #3

(1) Compute the limits:
(a) \( \lim_{z \to i} \frac{iz^3 - 1}{z^2 + 1} \);
(b) \( \lim_{z \to \infty} \frac{4z^2}{(z - 1)^2} \).

Solution. (a)
\[
\lim_{z \to i} \frac{iz^3 - 1}{z^2 + 1} = \lim_{z \to i} \frac{i(z^3 - i^3)}{z^2 + 1}
\]
\[
= \lim_{z \to i} \frac{i(z - i)(z^2 + iz + i^3)}{(z - i)(z + i)}
\]
\[
= \lim_{z \to i} \frac{iz^2 + iz + i^2}{z + i} = -\frac{3}{2}
\]
or by complex L'Hospital,
\[
\lim_{z \to i} \frac{iz^3 - 1}{z^2 + 1} = \lim_{z \to i} \frac{3iz^2}{2z} = -\frac{3}{2}
\]
(b)
\[
\lim_{z \to \infty} \frac{4z^2}{(z - 1)^2} = \lim_{z \to 0} \frac{4z^{-2}}{(z^{-1} - 1)^2}
\]
\[
= \lim_{z \to 0} \frac{4}{(1 - z)^2} = 4
\]

(2) Show that the limit of the function
\[
f(z) = \left( \frac{z}{\bar{z}} \right)^2
\]
as \( z \) tends to 0 does not exist.

Solution. Let \( z = x + yi \). When \( x \to 0 \) and \( y = 0 \),
\[
\lim_{x \to 0} \left( \frac{z}{\bar{z}} \right)^2 = \lim_{x \to 0} \frac{x^2}{x^2} = 1
\]
When \( x = y \) and \( x \to 0 \),
\[
\lim_{x \to 0} \left( \frac{z}{\bar{z}} \right)^2 = \lim_{x \to 0} \frac{(x + xi)^2}{(x - xi)^2} = -1
\]
Therefore, \( \lim_{z \to 0} f(z) \) does not exist.
(3) Let
\[ T(z) = \frac{az + b}{cz + d} \]
where \( ad - bc \neq 0 \). Show that
(a) \( \lim_{z \to \infty} T(z) = \infty \) if \( c = 0 \);
(b) \( \lim_{z \to \infty} T(z) = \frac{a}{c} \) if \( c \neq 0 \) and \( \lim_{z \to -d/c} T(z) = \infty \) if \( c \neq 0 \).

**Proof.** Since
\[
\lim_{z \to \infty} \frac{1}{T(z)} = \lim_{z \to 0} \frac{1}{T(z^{-1})} = \lim_{z \to 0} \frac{cz^{-1} + d}{az^{-1} + b} = \lim_{z \to 0} \frac{c + dz}{a + bz} = \frac{c}{a}
\]
it follows that \( \lim_{z \to \infty} T(z) = \infty \) when \( c = 0 \) and \( \lim_{z \to \infty} T(z) = \frac{a}{c} \) if \( c \neq 0 \).
Since \( ad - bc \neq 0 \), \( a(-d/c) + b \neq 0 \). Therefore,
\[
\lim_{z \to -d/c} \frac{1}{T(z)} = \lim_{z \to -d/c} \frac{cz + d}{az + b} = 0
\]
and hence \( \lim_{z \to -d/c} T(z) = \infty \) when \( c \neq 0 \). \( \square \)

(4) Find \( f'(z) \) when
(a) \( f(z) = 3z^2 - 2z + 4 \);
(b) \( f(z) = (1 - 4z^2)^3 \);
(c) \( f(z) = \frac{z - 1}{2z + 1} \) \( (z \neq -\frac{1}{2}) \);
(d) \( f(z) = \frac{(1 + z^2)^4}{z^2} \) \( (z \neq 0) \).

**Answer.** (a) \( 6z - 2 \) (b) \(-24z(1 - 4z^2)^2 \) (c) \( 3(2z + 1)^{-2} \) (d) \( 2(3z^2 - 1)(1 + z^2)^3z^{-3} \)

(5) Show that \( f'(z) \) does not exist at any point when
(a) \( f(z) = \text{Im}(z) \);
(b) \( f(z) = \begin{cases} z^2/z & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \)
Proof. (a) Since
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (x - yi) = 2 \neq 0
\]
for all \( z = x + yi \), \( f'(z) \) does not exist at any point.

(b) For \( z = x + yi \neq 0 \),
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \left( x - yi \right)^2
\]
\[
= \frac{2(x - yi)(x + yi) - (x - yi)^2}{(x + yi)^2}
\]
\[
+ i \frac{-2i(x - yi)(x + yi) - i(x - yi)^2}{(x + yi)^2}
\]
\[
= \frac{4(x - yi)}{x + yi} \neq 0
\]
Therefore, \( f''(z) \) does not exist for \( z \neq 0 \).
At \( z = 0 \),
\[
\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\pi^2}{z^2}
\]
When \( x \to 0 \) and \( y = 0 \),
\[
\lim_{x \to 0} \frac{\pi^2}{x^2} = \lim_{x \to 0} \frac{x^2}{x} = 1
\]
When \( x = y \) and \( x \to 0 \),
\[
\lim_{x \to 0} \frac{\pi^2}{x^2} = \lim_{x \to 0} \frac{(x - xi)^2}{(x + xi)^2} = -1
\]
Therefore, \( f'(0) \) does not exist. In conclusion, \( f'(z) \) does not exist anywhere. \( \square \)

(6) Use Cauchy-Riemann equations to verify that \( f(z) \) is analytic when
(a) \( f(z) = z^3 \) in \( \mathbb{C} \);
(b) \( f(z) = z^{-1} \) for \( z \neq 0 \);
(c) \( f(z) = e^{-z^2} \) in \( \mathbb{C} \).

Proof. (a) Since
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) z^3 = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (x + iy)^3
\]
\[
= 3(x + yi)^2 + i^23(x + yi)^2 = 0
\]
and $\partial f/\partial x$ and $\partial f/\partial y$ are continuous everywhere, $f(z)$ is entire.

(b) Since
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) z^{-1} = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (x + iy)^{-1} = -(x + yi)^{-2} - i^2(x + yi)^{-2} = 0
\]
and $\partial f/\partial x$ and $\partial f/\partial y$ are continuous for $z \neq 0$, $f(z)$ is analytic for $z \neq 0$.

(c) Since
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) e^{-z^2} = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) e^{-(x+iy)^2} = -2(x + yi)e^{-(x+iy)^2} - 2i^2(x + yi)e^{-(x+iy)^2} = 0
\]
and $\partial f/\partial x$ and $\partial f/\partial y$ are continuous everywhere, $f(z)$ is entire. \(\square\)

(7) Show that if both $f(z)$ and $g(z)$ satisfy the Cauchy-Riemann equations at $z_0$, so does $f(z)g(z)$.

Proof. Since C-R equations hold for $f(z)$ and $g(z)$ at $z_0$,
\[
\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} + i \frac{\partial g}{\partial y} = 0
\]
at $z = z_0$. Therefore,
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (fg) = \left( \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x} \right) + i \left( \frac{\partial f}{\partial y} g + f \frac{\partial g}{\partial y} \right) = g \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) + f \left( \frac{\partial g}{\partial x} + i \frac{\partial g}{\partial y} \right) = 0
\]
at $z_0$. Therefore, C-R equations hold for $f(z)g(z)$ at $z_0$. \(\square\)

(8) Suppose that $f(z) = u + iv$ is analytic at $z_0$. Show that
\[
f'(z_0) = -\frac{i}{z_0} \left( \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right)
\]
at $z = z_0$, where $(r, \theta)$ are the polar coordinates.
Proof. By chain rule,
\[ \frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}. \]

Since \( f(z) \) is analytic at \( z_0 \),
\[ \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f = 0 \]
and hence
\[ \left( \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right) f = 0 \Rightarrow \frac{\partial f}{\partial r} = -\frac{i}{r} \frac{\partial f}{\partial \theta} \]
in \( |z - z_0| < a \) for some \( a > 0 \). Therefore,
\[ f'(z) = \frac{\partial f}{\partial x} = \cos \theta \frac{\partial f}{\partial r} - \frac{\sin \theta}{r} \frac{\partial f}{\partial \theta} = -\frac{i}{r} \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta} - \frac{\sin \theta}{r} \frac{\partial f}{\partial \theta} = -\frac{i}{r} \left( \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right). \]

\[ \square \]