Math 311 Assignment #3
Due Oct. 1, 2010

(1) Compute the limits:
(a) \( \lim_{z \to i} \frac{iz^3 - 1}{z^2 + 1} \); 
(b) \( \lim_{z \to \infty} \frac{4z^2}{(z - 1)^2} \).

(2) Show that the limit of the function 
\( f(z) = \left( \frac{z}{i} \right)^2 \) 
as \( z \) tends to 0 does not exist.

(3) Let 
\( T(z) = \frac{az + b}{cz + d} \) 
where \( ad - bc \neq 0 \). Show that 
(a) \( \lim_{z \to \infty} T(z) = \infty \) if \( c = 0 \); 
(b) \( \lim_{z \to \infty} T(z) = \frac{a}{c} \) if \( c \neq 0 \) and \( \lim_{z \to -d/c} T(z) = \infty \) if \( c \neq 0 \).

(4) Find \( f'(z) \) when
(a) \( f(z) = 3z^2 - 2z + 4 \); 
(b) \( f(z) = (1 - 4z^2)^3 \); 
(c) \( f(z) = \frac{z - 1}{2z + 1} \) \( (z \neq -\frac{1}{2}) \); 
(d) \( f(z) = \frac{(1 + z^2)^4}{z^2} \) \( (z \neq 0) \).

(5) Show that \( f'(z) \) does not exist at any point when
(a) \( f(z) = \text{Im}(z) \); 
(b) \( f(z) = \begin{cases} \frac{z^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \).

(6) Use Cauchy-Riemann equations to verify that \( f(z) \) is analytic when
(a) \( f(z) = z^3 \) in \( \mathbb{C} \); 
(b) \( f(z) = z^{-1} \) for \( z \neq 0 \); 
(c) \( f(z) = e^{-z^2} \) in \( \mathbb{C} \).
(7) Show that if both $f(z)$ and $g(z)$ satisfy the Cauchy-Riemann equations at $z_0$, so does $f(z)g(z)$.

(8) Suppose that $f(z) = u + iv$ is analytic at $z_0$. Show that

$$f'(z_0) = -\frac{i}{z_0} \left( \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right)$$

at $z = z_0$, where $(r, \theta)$ are the polar coordinates.