(1) No books, notes or calculators are allowed. 
(2) Show your work in details.

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(1) (20 pts) Let $f(z)$ be the principal branch of $\sqrt[3]{z}$.

(a) (10 pts) Find $f(-i)$.

(b) (10 pts) Show that

$$f(z_1)f(z_2) = \lambda f(z_1z_2)$$

for all $z_1, z_2 \neq 0$, where $\lambda = 1, \frac{-1 + \sqrt{3}i}{2}$ or $\frac{-1 - \sqrt{3}i}{2}$. 
(2) (30 pts) Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function in a domain $D$, where $u(x, y) = \text{Re}(f(z))$ and $v(x, y) = \text{Im}(f(z))$. Show that if the function $u(x, y) - v(x, y)$ takes the minimum at a point in $D$, then $f(z)$ is constant in $D$. 
(3) (30 pts) Let
\[ f(z) = \frac{z^2}{z^2 - z - 2} \]
Find the Laurent series of \( f(z) \) in each of the following domains:
(a) (15 pts) \( 1 < |z| < 2 \)
(b) (15 pts) \( 0 < |z - 2| < 1 \)
(4) (40 pts) Do the following:

(a) (10 pts) Find \( \cos \left( \frac{\pi}{3} + i \right) \).

(b) (10 pts) Find the Taylor series of \((\cos z)^2\) at \(z = \pi\).
(c) (10 pts) Show that
\[ |\cos(z)|^2 = (\cos x)^2 + (\sinh y)^2 \]
for all \( z \in \mathbb{C} \), where \( x = \text{Re}(z) \) and \( y = \text{Im}(z) \).

(d) (10 pts) Let \( C_N \) be the boundary of the square
\[ \{ |x| \leq N\pi, |y| \leq N\pi \}, \]
where \( N \) is a positive integer. Show that
\[ \lim_{N \to \infty} \int_{C_N} \frac{dz}{z^3 \cos z} = 0 \]
(5) (20 pts) Let $C$ be the circle $|z| = 1$ oriented counter-clockwise.

(a) (10 pts) Compute
$$\int_{C} \frac{1}{z^2 - 8z + 1} \, dz$$

(b) (10 pts) Use or not use part (a) to compute
$$\int_{0}^{\pi} \frac{1}{4 - \cos \theta} \, d\theta$$
(6) (40 pts) For each of the following complex functions, do the following:

• find all its singularities in $\mathbb{C}$;
• write the principal part of the function at each singularity;
• for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
• compute the residue of the function at each singularity.

(a) (10 pts) $f(z) = (1 - z^2) \exp \left( \frac{1}{z} \right)$

(b) (10 pts) $f(z) = \frac{1}{(\sin z)^2}$
(c) (10 pts) \( f(z) = \frac{1 - \cos z}{z^2} \)

(d) (10 pts) \( f(z) = \frac{e^z}{z(z - 1)^2} \)
(7) (20 pts) Compute the following contour integrals.

(a) (10 pts)
\[ \int_{L} z dz, \]
where \( L \) is the boundary of the triangle \( ABC \) with \( A = 0 \), \( B = 1 \) and \( C = i \), oriented counter-clockwise.

(b) (10 pts)
\[ \int_{C} \frac{z^{2008}}{z^{2009} + z + 1} dz \]
where \( C \) is the circle \( |z| = 2 \) oriented counter-clockwise.