Math 309 (A1) HW 4 Solution

A1. We integrate the complex function

\[ f(z) = \frac{1}{1 + z^4} \]

along the line \( L_R = \{ -R \leq \text{Re}(z) \leq R, \text{Im}(z) = 0 \} \) and the semicircle \( C_R = \{|z| = R, \text{Im}(z) \geq 0 \} \), going counter-clockwise. Note that \( 1 + z^4 = 0 \) has four roots, given by

\[ z_k = \sqrt[4]{-1} = \sqrt[4]{e^{2k+1}\pi j} = e^{(2k+1)\pi j/4} \]

for \( k = 0, 1, 2, 3 \). Obviously, when \( R \) is large enough, two of the roots, \( z_0 \) and \( z_1 \), lie inside the contour above. By CIF,

\[
\int_{L_R} f(z)dz + \int_{C_R} f(z)dz = \int_{|z-z_0|=r} f(z)dz + \int_{|z-z_1|=r} f(z)dz
\]

where \( |z-z_0| = r \) and \( |z-z_1| = r \) are two small circles centered at \( z_0 \) and \( z_1 \), respectively. When we take the limit as \( R \to \infty \), we see that

\[
\lim_{R \to \infty} \int_{L_R} f(z)dz = \int_{-\infty}^{\infty} \frac{1}{1 + x^4} dx
\]

and

\[
\lim_{R \to \infty} \int_{C_R} f(z)dz = \lim_{R \to \infty} \int_{0}^{\pi} f(Re^{jt})Re^{jt} dt
\]

\[
= j \lim_{R \to \infty} \int_{0}^{\pi} \left( \frac{Re^{jt}}{1 + R^4e^{4jt}} \right) dt
\]

\[
= j \lim_{R \to \infty} \int_{0}^{\pi} \left( \frac{e^{jt}}{1/R^3 + e^{4jt}} \right) dt = 0
\]

Therefore, we have

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^4} dx = \int_{|z-z_0|=r} f(z)dz + \int_{|z-z_1|=r} f(z)dz
\]

Note that

\[
\int_{|z-z_0|=r} f(z)dz = \int_{|z-z_0|=r} \frac{1}{(z-z_0)(z-z_1)(z-z_2)(z-z_3)} dz
\]

\[
= \int_{|z-z_0|=r} \frac{g(z)}{z-z_0} dz
\]

where

\[
g(z) = \frac{1}{(z-z_1)(z-z_2)(z-z_3)}
\]
By CIF (Version II),
\[ \int_{|z - z_0| = r} f(z) \, dz = \int_{|z - z_0| = r} \frac{g(z)}{z - z_0} \, dz = 2\pi j g(z_0) \]
\[ = \frac{2\pi j}{(z_0 - z_1)(z_0 - z_2)(z_0 - z_3)} \]
By the same argument,
\[ \int_{|z - z_1| = r} f(z) \, dz = \frac{2\pi j}{(z_1 - z_0)(z_1 - z_2)(z_1 - z_3)} \]
Therefore,
\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^4} \, dx = \frac{2\pi j}{(z_0 - z_1)(z_0 - z_2)(z_0 - z_3)} + \frac{2\pi j}{(z_1 - z_0)(z_1 - z_2)(z_1 - z_3)} \]
Plugging in \( z_0 = \sqrt{2}/2(1+j) \), \( z_1 = \sqrt{2}/2(-1+j) \), \( z_2 = \sqrt{2}/2(-1-j) \) and \( z_3 = \sqrt{2}/2(1-j) \), we have
\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^4} \, dx = \frac{\sqrt{2}}{2\pi} \]

A2. Let \( f(z) = u(x, y) + jv(x, y) \). Since \( f(z) \) takes real values for \( z \) real, i.e., when \( \text{Im}(z) = y = 0, v(x, 0) = 0 \) for all \( x \). Therefore, \( v_x(x, 0) = 0 \) for all \( x \). On the other hand, since \( f(z) \) is analytic,
\[ f'(z) = u_x(x, y) + jv_x(x, y) \]
Hence, \( v_x(x, 0) = 0 \Rightarrow f'(z) \) takes real values when \( z \) is real. By induction, this is true for all \( f^{(n)}(z) \).