A3.1 Let \( P_2 = \{ f(x) \in \mathbb{R}[x] : \deg f \leq 2 \} \) be the vector space of polynomials in \( x \) of degree \( \leq 2 \) and let \( T_1 : P_2 \to P_2 \) and \( T_2 : P_2 \to P_2 \) be two linear transformations given by
\[
T_1(f(x)) = f(x + 1) - f(x) \quad \text{and} \quad T_2(f(x)) = f(1 - x).
\]
(a) Find the matrix representations \([T_1 + T_2]_{D \leftarrow D}\), \([T_1 \circ T_2]_{D \leftarrow D}\) and \([T_2 \circ T_1]_{D \leftarrow D}\) of \( T_1 + T_2, T_1 \circ T_2 \) and \( T_2 \circ T_1 \), respectively, under the standard basis \( D = \{1, x, x^2\} \) of \( P_2 \).
(b) Find the kernels, ranges and ranks of \( T_1 + T_2, T_1 \circ T_2 \) and \( T_2 \circ T_1 \), respectively.
(c) Verify Rank Theorem for \( T_1 + T_2, T_1 \circ T_2 \) and \( T_2 \circ T_1 \), respectively.

A3.2 Let \( T_1 : V \to W \) and \( T_2 : V \to W \) be two linear transformations between finite-dimensional vector spaces \( V \) and \( W \). Do the following:
(a) Prove that \( R(T_1 + T_2) \subset R(T_1) + R(T_2) \).
(b) Prove that \( \text{rank}(T_1 + T_2) \leq \text{rank}(T_1) + \text{rank}(T_2) \).
(c) Give an example of \( T_1 \) and \( T_2 \) such that
\[
\text{rank}(T_1 + T_2) = \text{rank}(T_1) + \text{rank}(T_2)
\]
and give an example of \( T_1 \) and \( T_2 \) such that
\[
\text{rank}(T_1 + T_2) < \text{rank}(T_1) + \text{rank}(T_2).
\]

A3.3 Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear transformation satisfying
\[
T(e_2) = e_1, \quad T(e_3) = e_2 \quad \text{and} \quad T^3 = 0.
\]
(a) Find the matrix \([T]_{D \leftarrow D}\) representing \( T \) under the standard basis \( D \) of \( \mathbb{R}^3 \).
(b) Find the kernel \( K(T) \) and the range \( R(T) \) of \( T \).

A3.4 Determine whether the following linear transformations are 1-1, onto and/or bijective/invertible. Justify your answer:
(a) \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) given by \( T(x, y) = (x + y, x + 2y, 2x + y) \).
(b) \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) given by \( T(x, y, z) = (x + y + z, x - y + z) \).
(c) \( T : P_3 \to \mathbb{R}^4 \) given by \( T(f(x)) = (f(0), f(1), f(2), f(3)) \),
where \( P_3 \) is the vector space of polynomials \( f(x) \in \mathbb{R}[x] \) of degree \( \leq 3 \).
(d) $T : M_{3\times 3}(\mathbb{R}) \to M_{3\times 3}(\mathbb{R})$ given by $T(A) = BA$, where

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 \\ 1 \end{bmatrix}.$$

A3.5 Let $T : V \to W$ and $S : U \to V$ be two linear transformations between vector spaces $U, V$ and $W$.

(a) Show that $T \circ S = 0$ if and only if $R(S) \subset K(T)$.

(b) Suppose that $\dim V < \infty$. Show that if $T \circ S = 0$, then $\text{rank}(T) + \text{rank}(S) \leq \dim V$. 