A1.1 Let $M_{3 \times 3}(\mathbb{R})$ be the vector space of all $3 \times 3$ real matrices. Determine whether the following sets are subspaces of $M_{3 \times 3}(\mathbb{R})$. Justify your answer.

(a) All $3 \times 3$ upper-triangular real matrices.
(b) All $3 \times 3$ singular real matrices.
(c) All $3 \times 3$ real matrices whose first and second rows are identical.
(d) All $3 \times 3$ real matrices $A$ satisfying

$$A \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} A.$$ 

A1.2 Let $P_2$ be the vector space of real polynomials $f(x)$ in $x$ of $\deg f(x) \leq 2$ and let

$V_1 = \{ f(x) \in P_2 : f(x) \equiv f(1-x) \}$ and $V_2 = \{ f(x) \in P_2 : f(1) = 0 \}$.

Do the following:

(a) Show that $V_1$ and $V_2$ are subspaces of $P_2$.
(b) Find $V_1 \cap V_2$ and $V_1 + V_2$.

A1.3 Let $A$ and $B$ be two $n \times n$ matrices satisfying $AB = BA$. Show that $\text{Nul}(A) + \text{Nul}(B) \subset \text{Nul}(AB)$.

A1.4 Construct the following examples with the required properties. You must justify your answer.

(a) A vector space $V$ over $\mathbb{R}$ with only two subspaces.
(b) Two subsets $S_1$ and $S_2$ of a vector space $V$ over $\mathbb{R}$ such that $\text{Span}(S_1) \subset \text{Span}(S_2)$ but $S_1 \not\subset S_2$.
(c) Two $2 \times 2$ matrices $A$ and $B$ such that $\text{Nul}(A) = \text{Nul}(B)$, $\text{Col}(A) = \text{Col}(B)$ but $A \neq B$.
(d) Two subspaces $W_1$ and $W_2$ of $\mathbb{R}^2$ such that $W_1 \cup W_2$ is also a subspace.

A1.5 Let $W_1, W_2, W_3$ be three subspaces of a vector space $V$. Show that if $W_1 \subset W_2 \cup W_3$, then either $W_1 \subset W_2$ or $W_1 \subset W_3$. 

1