Outline

1. Matrix Representation of Linear Transformations
2. Spaces of Linear Transformations
3. Kernel and Range
Matrix Representations of $T_1 + T_2$ and $T_1 \circ T_2$

Let $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ be two linear transformations from $V \rightarrow W$ and let $B$ and $C$ be two ordered bases of $V$ and $W$, respectively. Then for all $c \in \mathbb{R}$,

$$[T_1 + T_2]_{C \leftarrow B} = [T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B} \quad \text{and} \quad [cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}$$

Let $T_1 : V \rightarrow W$ and $T_2 : U \rightarrow V$ be two linear transformations and let $B$, $C$, $D$ be three ordered bases of $U$, $V$, $W$, respectively. Then

$$[T_1 \circ T_2]_{D \leftarrow B} = [T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B}$$
Let $T_1 : V \to W$ and $T_2 : V \to W$ be two linear transformations from $V \to W$ and let $B$ and $C$ be two ordered bases of $V$ and $W$, respectively. Then for all $c \in \mathbb{R}$,

$$[T_1 + T_2]_{C \leftarrow B} = [T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B} \quad \text{and} \quad [cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}$$

Let $T_1 : V \to W$ and $T_2 : U \to V$ be two linear transformations and let $B, C, D$ be three ordered bases of $U, V, W$, respectively. Then

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Let $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ be two linear transformations from $V \rightarrow W$ and let $B$ and $C$ be two ordered bases of $V$ and $W$, respectively. Then for all $c \in \mathbb{R}$,

$$[T_1 + T_2]_{C \leftarrow B} = [T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B} \text{ and } [cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}$$

Let $T_1 : V \rightarrow W$ and $T_2 : U \rightarrow V$ be two linear transformations and let $B, C, D$ be three ordered bases of $U, V, W$, respectively. Then

$$[T_1 \circ T_2]_{D \leftarrow B} = [T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B}$$
Proofs for \([ T_1 + T_2 ]\) and \([ cT_1 ]\)

For \( T_1 : V \rightarrow W, T_2 : V \rightarrow W \) and \( v \in V \),

\[
[(T_1 + T_2)(v)]_C = [T_1 + T_2]_{C \leftarrow B}[v]_B
\]

On the other hand,

\[
[(T_1 + T_2)(v)]_C = [T_1(v) + T_2(v)]_C = [T_1(v)]_C + [T_2(v)]_C
\]

\[
= [T_1]_{C \leftarrow B}[v]_B + [T_2]_{C \leftarrow B}[v]_B
\]

\[
= ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})[v]_B
\]

So \([ T_1 + T_2 ]_{C \leftarrow B} = ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})\).

Similarly, Since

\[
[cT_1(v)]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}[v]_B
\]

\[
[cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}.
\]
Proofs for \([T_1 + T_2]\) and \([cT_1]\)

For \(T_1 : V \rightarrow W\), \(T_2 : V \rightarrow W\) and \(v \in V\),

\[
[(T_1 + T_2)(v)]_C = [T_1 + T_2]_{C \leftarrow B}[v]_B
\]

On the other hand,

\[
[(T_1 + T_2)(v)]_C = [T_1(v) + T_2(v)]_C = [T_1(v)]_C + [T_2(v)]_C
\]

\[
= [T_1]_{C \leftarrow B}[v]_B + [T_2]_{C \leftarrow B}[v]_B
\]

\[
= ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})[v]_B
\]

So \([T_1 + T_2]_{C \leftarrow B} = ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})\).

Similarly, Since

\[
[cT_1(v)]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}[v]_B
\]

\[
[cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}.
\]
Proofs for $[T_1 + T_2]$ and $[cT_1]$

For $T_1 : V \rightarrow W$, $T_2 : V \rightarrow W$ and $v \in V$,

$$[(T_1 + T_2)(v)]_C = [T_1 + T_2]_{C \leftarrow B}[v]_B$$

On the other hand,

$$[(T_1 + T_2)(v)]_C = [T_1(v) + T_2(v)]_C = [T_1(v)]_C + [T_2(v)]_C$$

$$= [T_1]_{C \leftarrow B}[v]_B + [T_2]_{C \leftarrow B}[v]_B$$

$$= ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})[v]_B$$

So $[T_1 + T_2]_{C \leftarrow B} = ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})$.

Similarly, Since

$$[cT_1(v)]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}[v]_B$$

$$[cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}.$$
Proofs for \([T_1 + T_2]\) and \([cT_1]\)

For \(T_1 : V \to W, T_2 : V \to W\) and \(v \in V\),

\[
[(T_1 + T_2)(v)]_C = [T_1 + T_2]_{C \leftarrow B}[v]_B
\]

On the other hand,

\[
[(T_1 + T_2)(v)]_C = [T_1(v) + T_2(v)]_C = [T_1(v)]_C + [T_2(v)]_C
\]

\[
= [T_1]_{C \leftarrow B}[v]_B + [T_2]_{C \leftarrow B}[v]_B
\]

\[
= ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})[v]_B
\]

So \([T_1 + T_2]_{C \leftarrow B} = ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B}).\)

Similarly, Since

\[
[cT_1(v)]_C = c[T_1]_{C \leftarrow B}[v]_B
\]

\[
[cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}.
\]
Proofs for \([T_1 \circ T_2]\)

For \(T_1 : V \rightarrow W\), \(T_2 : U \rightarrow V\) and \(u \in U\),

\[
[(T_1 \circ T_2)(u)]_D = [T_1 \circ T_2]_{D \leftarrow B}[u]_B
\]

On the other hand,

\[
[(T_1 \circ T_2)(u)]_D = [T_1(T_2(u))]_D = [T_1]_{D \leftarrow C}[T_2(u)]_C
\]

\[
= [T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B}[u]_B
\]

So \([T_1 \circ T_2]_{D \leftarrow B} = ([T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B}).\)
Proofs for $[T_1 \circ T_2]$

For $T_1 : V \rightarrow W$, $T_2 : U \rightarrow V$ and $u \in U$,

$$[(T_1 \circ T_2)(u)]_D = [T_1 \circ T_2]_{D \leftarrow B}[u]_B$$

On the other hand,

$$[(T_1 \circ T_2)(u)]_D = [T_1(T_2(u))]_D = [T_1]_{D \leftarrow C}[T_2(u)]_C$$
$$= [T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B}[u]_B$$

So $[T_1 \circ T_2]_{D \leftarrow B} = ([T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B})$. 
Proofs for \([T_1 \circ T_2]\)

For \(T_1 : V \rightarrow W, \ T_2 : U \rightarrow V\) and \(u \in U\),

\[
[(T_1 \circ T_2)(u)]_D = [T_1 \circ T_2]_{D \leftarrow B}[u]_B
\]

On the other hand,

\[
[(T_1 \circ T_2)(u)]_D = [T_1(T_2(u))]_D = [T_1]_{D \leftarrow C}[T_2(u)]_C
\]

\[
= [T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B}[u]_B
\]

So \([T_1 \circ T_2]_{D \leftarrow B} = ([T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B})\).
Let $T : V \to W$ be a linear transformation, let $B$ and $B'$ be two ordered bases of $V$ and let $C$ and $C'$ be two ordered bases of $W$. Then

$$[T]_{C' \leftarrow B'} = P_{C' \leftarrow C}[T]_{C \leftarrow B} P_{B \leftarrow B'}$$

Let $I_V : V \to V$ and $I_W : W \to W$ be the identity maps on $V$ and $W$. Then

$$[I_W \circ T \circ I_V]_{C' \leftarrow B'} = [I_W]_{C' \leftarrow C}[T]_{C \leftarrow B}[I_V]_{B \leftarrow B'}$$

$$= P_{C' \leftarrow C}[T]_{C \leftarrow B} P_{B \leftarrow B'}$$
Let $T : V \rightarrow W$ be a linear transformation, let $B$ and $B'$ be two ordered bases of $V$ and let $C$ and $C'$ be two ordered bases of $W$. Then

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$$[I_W \circ T \circ I_V]_{C' \leftarrow B'} = [I_W]_{C' \leftarrow C}[T]_{C \leftarrow B}[I_V]_{B \leftarrow B'}$$

$$= P_{C' \leftarrow C}[T]_{C \leftarrow B}P_{B \leftarrow B'}$$
Theorem

- For all linear transformations $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ and $c \in \mathbb{R}$, $T_1 + T_2$ and $cT_1$ are also linear transformations from $V$ to $W$.
- Furthermore,

$$[T_1 + T_2]_{B_2 \leftarrow B_1} = [T_1]_{B_2 \leftarrow B_1} + [T_2]_{B_2 \leftarrow B_1} \quad \text{and}$$

$$[cT_1]_{B_2 \leftarrow B_1} = c[T_1]_{B_2 \leftarrow B_1}$$

where $B_1$ is a basis for $V$ and $B_2$ is a basis for $W$.
- Let $L(V, W)$ be the set of all linear transformations from $V$ to $W$. Then $L(V, W)$ is itself a vector space over $\mathbb{R}$ under the addition and scalar multiplication defined above.
For all linear transformations $T_1 : V \to W$ and $T_2 : V \to W$ and $c \in \mathbb{R}$, $T_1 + T_2$ and $cT_1$ are also linear transformations from $V$ to $W$.

Furthermore,

$$[T_1 + T_2]_{B_2 \leftarrow B_1} = [T_1]_{B_2 \leftarrow B_1} + [T_2]_{B_2 \leftarrow B_1} \quad \text{and} \quad [cT_1]_{B_2 \leftarrow B_1} = c[T_1]_{B_2 \leftarrow B_1}$$

where $B_1$ is a basis for $V$ and $B_2$ is a basis for $W$. Let $L(V, W)$ be the set of all linear transformations from $V$ to $W$. Then $L(V, W)$ is itself a vector space over $\mathbb{R}$ under the addition and scalar multiplication defined above.
Theorem

For all linear transformations $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ and $c \in \mathbb{R}$, $T_1 + T_2$ and $cT_1$ are also linear transformations from $V$ to $W$.

Furthermore,

$$[T_1 + T_2]_{B_2 \leftarrow B_1} = [T_1]_{B_2 \leftarrow B_1} + [T_2]_{B_2 \leftarrow B_1} \quad \text{and} \quad [cT_1]_{B_2 \leftarrow B_1} = c[T_1]_{B_2 \leftarrow B_1}$$

where $B_1$ is a basis for $V$ and $B_2$ is a basis for $W$.

Let $L(V, W)$ be the set of all linear transformations from $V$ to $W$. Then $L(V, W)$ is itself a vector space over $\mathbb{R}$ under the addition and scalar multiplication defined above.
Isomorphism Between $L(V, W)$ and $M_{m \times n}(\mathbb{R})$

Let $V$ and $W$ be two vectors spaces over $\mathbb{R}$ of dimensions $\dim V = n$ and $\dim W = m$. Fixing two ordered bases $B$ of $V$ and $C$ of $W$, there is a map

$$F : L(V, W) \rightarrow M_{m \times n}(\mathbb{R})$$

given by $F(T) = [T]_{C \leftarrow B}$.

Then $F$ is an invertible linear map. We say $L(V, W) \cong M_{m \times n}(\mathbb{R})$.

For example,

$$L(\mathbb{R}^3, \mathbb{R}^2) \cong M_{2 \times 3}(\mathbb{R})$$
$$L(P_4, P_5) \cong M_{6 \times 5}(\mathbb{R})$$
$$L(M_{3 \times 4}(\mathbb{R}), M_{2 \times 3}(\mathbb{R})) \cong M_{6 \times 12}(\mathbb{R})$$
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$$L(M_{3 \times 4}(\mathbb{R}), M_{2 \times 3}(\mathbb{R})) \cong M_{6 \times 12}(\mathbb{R})$$
**Definition**

Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. The *kernel* of $T$ is $K(T) = \ker(T) = \{ x \in V : T(x) = 0 \} \subset V$. The *range* of $T$ is the image of $T$, i.e., $R(T) = T(V) = \{ T(x) : x \in V \} \subset W$.

**Theorem**

Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. Then $K(T)$ is a subspace of $V$ and $R(T)$ is a subspace of $W$.

Let $T(x, y) = (x, x)$ be a linear transformation from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Then $K(T) = \{ (x, y) : T(x, y) = (0, 0) \} = \{ (x, y) : x = 0 \}$ and $R(T) = \{ T(x, y) \} = \{ (x, x) \} = \{ (x, y) : x - y = 0 \}$. 
Definition

Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. The \textit{kernel} of $T$ is $K(T) = \ker(T) = \{ x \in V : T(x) = 0 \} \subset V$. The \textit{range} of $T$ is the image of $T$, i.e., $R(T) = T(V) = \{ T(x) : x \in V \} \subset W$.

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Kernel and Range

**Definition**

Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. The *kernel* of $T$ is $K(T) = \ker(T) = \{x \in V : T(x) = 0\} \subset V$. The *range* of $T$ is the image of $T$, i.e.,

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**Theorem**

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Definition

Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. The \textit{kernel} of $T$ is $K(T) = \ker(T) = \{ x \in V : T(x) = 0 \} \subset V$. The \textit{range} of $T$ is the image of $T$, i.e., $R(T) = T(V) = \{ T(x) : x \in V \} \subset W$.

Theorem

Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. Then $K(T)$ is a subspace of $V$ and $R(T)$ is a subspace of $W$.

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Proof that $K(T)$ and $R(T)$ are subspaces

**$K(T)$ is a subspace of $V$.**

Since $T(0) = 0$, $0 \in K(T)$. For all $v_1, v_2 \in K(T)$, $T(v_1) = T(v_2) = 0$ and hence

$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = 0$$

for all $c \in \mathbb{R}$. Therefore, $v_1 + cv_2 \in K(T)$.

**$R(T)$ is a subspace of $W$.**

Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Thus,

$$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in R(T).$$
Proof that $K(T)$ and $R(T)$ are subspaces

**$K(T)$ is a subspace of $V$.**

Since $T(0) = 0$, $0 \in K(T)$. For all $v_1, v_2 \in K(T)$, $T(v_1) = T(v_2) = 0$ and hence

$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = 0$$

for all $c \in \mathbb{R}$. Therefore, $v_1 + cv_2 \in K(T)$.

**$R(T)$ is a subspace of $W$.**

Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Thus,

$$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in R(T).$$
Proof that $K(T)$ and $R(T)$ are subspaces

**$K(T)$ is a subspace of $V$.**

Since $T(0) = 0$, $0 \in K(T)$. For all $\mathbf{v}_1, \mathbf{v}_2 \in K(T)$, $T(\mathbf{v}_1) = T(\mathbf{v}_2) = 0$ and hence

$$T(\mathbf{v}_1 + c\mathbf{v}_2) = T(\mathbf{v}_1) + cT(\mathbf{v}_2) = 0$$

for all $c \in \mathbb{R}$. Therefore, $\mathbf{v}_1 + c\mathbf{v}_2 \in K(T)$.

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**$R(T)$ is a subspace of $W$.**

Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $w_1 = T(\mathbf{v}_1)$ and $w_2 = T(\mathbf{v}_2)$. Thus,

$$w_1 + cw_2 = T(\mathbf{v}_1) + cT(\mathbf{v}_2) = T(\mathbf{v}_1 + c\mathbf{v}_2) \in R(T).$$
Proof that $K(T)$ and $R(T)$ are subspaces

$K(T)$ is a subspace of $V$.

Since $T(0) = 0$, $0 \in K(T)$. For all $v_1, v_2 \in K(T)$, $T(v_1) = T(v_2) = 0$ and hence

$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = 0$$

for all $c \in \mathbb{R}$. Therefore, $v_1 + cv_2 \in K(T)$.

$R(T)$ is a subspace of $W$.

Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Thus,

$$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in R(T).$$
Range and Rank

If \( R(T) \) is finite-dimensional, then the dimension of \( R(T) \) is called the \textit{rank} of \( T \), denoted by

\[
\text{rank}(T) = \dim R(T) = \dim T(V).
\]

Given a basis \( B = \{v_1, v_2, ..., v_n\} \) of \( V \), then the range of a linear transformation \( T : V \rightarrow W \) is

\[
R(T) = T(V) = \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\}.
\]

Note that

\[
\text{rank}(T) = \dim R(T) = \dim \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\} \leq n = \dim V.
\]
Range and Rank

If $R(T)$ is finite-dimensional, then the dimension of $R(T)$ is called the *rank* of $T$, denoted by

$$\text{rank}(T) = \dim R(T) = \dim T(V).$$

Given a basis $B = \{v_1, v_2, ..., v_n\}$ of $V$, then the range of a linear transformation $T : V \to W$ is

$$R(T) = T(V) = \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\}.$$ 

Note that

$$\text{rank}(T) = \dim R(T) = \dim \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\} \leq n = \dim V.$$
Range and Rank

If $R(T)$ is finite-dimensional, then the dimension of $R(T)$ is called the \textit{rank} of $T$, denoted by

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Given a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $V$, then the range of a linear transformation $T : V \to W$ is

$$R(T) = T(V) = \text{Span}\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}.$$ 

Note that

$$\text{rank}(T) = \dim R(T) = \dim \text{Span}\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\} \leq n = \dim V.$$
Kernel, Range and Rank of $T : \mathbb{R}^n \to \mathbb{R}^m$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by

$$T(v) = Av$$

for an $m \times n$ matrix $A$. Then

$$K(T) = \{v : T(v) = 0\} = \{v : Av = 0\} = \text{Nul}(A).$$

$$R(T) = \text{Span}\{T(e_1), T(e_2), \ldots, T(e_n)\}$$
$$= \text{Span}\{Ae_1, Ae_2, \ldots, Ae_n\} = \text{Col}(A).$$

$$\text{rank}(T) = \dim R(T) = \dim \text{Col}(A) = \text{rank}(A).$$
Kernel, Range and Rank of $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$

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$$K(T) = \{v : T(v) = 0\} = \{v : Av = 0\} = \text{Nul}(A).$$

$$R(T) = \text{Span}\{T(e_1), T(e_2), \ldots, T(e_n)\}$$

$$= \text{Span}\{Ae_1, Ae_2, \ldots, Ae_n\} = \text{Col}(A).$$

$$\text{rank}(T) = \dim R(T) = \dim \text{Col}(A) = \text{rank}(A).$$
Kernel, Range and Rank of $T : \mathbb{R}^n \to \mathbb{R}^m$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by

$$T(\mathbf{v}) = A\mathbf{v}$$

for an $m \times n$ matrix $A$. Then

$$K(T) = \{\mathbf{v} : T(\mathbf{v}) = 0\} = \{\mathbf{v} : A\mathbf{v} = 0\} = \text{Nul}(A).$$

$$R(T) = \text{Span}\{T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)\}$$

$$= \text{Span}\{A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_n\} = \text{Col}(A).$$

$$\text{rank}(T) = \dim R(T) = \dim \text{Col}(A) = \text{rank}(A).$$