(1) No books, notes or calculators are allowed.
(2) Show your work in details.

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Notations and formulas:

- \( \mathbb{R} \) is the set of real numbers.
- \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space.
- \( \mathbb{R}[x] \) is the set of polynomials in \( x \) with real coefficients.
- \( M_{m \times n}(\mathbb{R}) \) is the set of \( m \times n \) matrices with real entries.
- For a vector space \( V \) and an ordered basis \( B \) of \( V \), \( [v]_B \) is the coordinate vector of a vector \( v \in V \) under \( B \).
- For two ordered bases \( B = \{v_1, v_2, ..., v_n\} \) and \( C = \{u_1, u_2, ..., u_n\} \) of a vector space \( V \), the change-of-basis matrix \( P_{B \rightarrow C} \) is given by
  \[
P_{B \rightarrow C} = \begin{bmatrix}
    [u_1]_B & [u_2]_B & \cdots & [u_n]_B
  \end{bmatrix}.
  \]
(1) (20 points) Determine whether the following sets $W$ are subspaces of the given vector spaces $V$. Justify your answer.

(a) (10 points) $V = \mathbb{R}^2$ and $W = \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$.

(b) (10 points) $V = \mathbb{R}[x]$ and $W = \{f(x) \in \mathbb{R}[x] : f(1) = f(2) + f(3)\}$. 
(2) (30 points) Let
\[ B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \]
and \( V = \{ A \in M_{2 \times 2}(\mathbb{R}) : AB = 0 \} \). Do the following:

(a) (10 points) Show that \( V \) is a subspace of \( M_{2 \times 2}(\mathbb{R}) \).
(b) (20 points) Find dim $V$ and a basis for $V$. 
(3) (20 points) Determine whether the following maps $T$ are linear transformations. Justify your answer.

(a) (10 points) $T : M_{n \times n}(\mathbb{R}) \to M_{n \times n}(\mathbb{R})$ given by $T(A) = 2A - A^T$.

(b) (10 points) $T : \mathbb{R}[x] \to \mathbb{R}$ given by $T(f(x)) = f(1)f(2)$. 
(4) (30 points) Let $V = \{ f(x) \in \mathbb{R}[x] : \deg f(x) \leq 2 \}$ and let

$B = \{ 1, x + 1, (x + 1)^2 \}$ and $C = \{ 1, 1 - x, (1 - x)^2 \}$

be two ordered bases of $V$. Do the following:

(a) (20 points) Find the change-of-basis matrices $P_{B \leftarrow C}$ and $P_{C \leftarrow B}$.

(b) (10 points) Let $f(x)$ be a polynomial in $V$ satisfying

$[f(x)]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

Find $[f(x)]_C$. 