Math 225 Final Review

Some information on the final:

- Time and location: 09:00-11:00, Tuesday, December 13, 2016, CCIS L1140
- Sections covered by the final: (Poole’s Book) 6.1-6.6, 4.1-4.4, 4.6 (p. 335-348), 5.1, 5.2, 5.3 (p. 388-392), 5.4

A list of topics covered by the final:

- Vector Space
- Subspace
- Null, Row and Column Spaces
- Linear Dependence
- Span, Basis and Dimension
- Change of Basis
- Linear Transformation
- Matrix Representation of Linear Transformation
- Kernel and Range of Linear Transformation
- Injectivity, Surjectivity and Bijectivity of Linear Transformation
- Rank and Rank Theorem
- Eigenvalue, Eigenvector and Characteristic Polynomial
- Diagonalization
- Applications of Diagonalization: Linear Recurrence and Systems of Linear ODEs
- Inner Product, Norm and Orthogonality
- Orthogonal Complement and Orthogonal Projection
- Gram-Schmidt Process
- Orthogonal Diagonalization and Spectral Decomposition of Symmetric Matrices

Important Theorems:

- Rank Theorem: \( \dim K(T) + R(T) = \dim V \) for a linear transformation \( T : V \to W \).
- Diagonalization Theorem: An \( n \times n \) matrix \( A \) (a linear endomorphism \( T : V \to V \) on a vector space of dimension \( n \)) is diagonalizable if and only if \( A \) (\( T \)) has \( n \) linearly independent eigenvectors.
- Spectral Theorem: Every real symmetric matrix is orthogonally diagonalizable.
Important Algorithms and Formulas:

- Compute \([v]_B\) for a vector \(v \in V\) and an ordered basis \(B\) of \(V\).
- Compute \(P_{C \leftarrow B}\) for two ordered bases \(B\) and \(C\) of \(V\).
- Change-of-basis formula:
  \[ P_{D \leftarrow B} = P_{D \leftarrow C}P_{C \leftarrow B}. \]
- Compute \(\text{rank}(A), \text{Null}(A), \text{Row}(A), \text{Col}(A), \text{rank}(T), K(T)\) and \(R(T)\) for matrices \(A\) and linear transformations \(T\).
- Compute \([T]_{C \leftarrow B}\) for a linear transformation \(T : V \to W\), an ordered basis \(B\) for \(V\) and an ordered basis \(C\) for \(W\).
- Change-of-basis for \([T]\):
  \[ [T]_{C' \leftarrow B'} = P_{C' \leftarrow C} [T]_{C \leftarrow B} P_{B' \leftarrow B}. \]
- Compute characteristic polynomials and eigenvalues, eigenvectors of square matrices and linear endomorphisms.
- Diagonalize square matrices and linear endomorphisms:
  - If an \(n \times n\) matrix \(A\) has \(n\) linear independent eigenvectors \(v_1, v_2, ..., v_n\) with corresponding eigenvalues \(\lambda_1, \lambda_2, ..., \lambda_n\), then
    \[ P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix} \text{ for } P = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix} \]
  - If a linear endomorphism \(T : V \to V\) on a vector space \(V\) of dimension \(n\) has \(n\) linear independent eigenvectors \(v_1, v_2, ..., v_n\) with corresponding eigenvalues \(\lambda_1, \lambda_2, ..., \lambda_n\), then
    \[ [T]_{D \leftarrow D} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix} \text{ for } D = \{v_1, v_2, ..., v_n\} \]
- Compute \(A^m\) and \(T^m\):
  - If an \(n \times n\) matrix \(A\) has \(n\) linearly independent eigenvectors \(v_1, v_2, ..., v_n\) with corresponding eigenvalues \(\lambda_1, \lambda_2, ..., \lambda_n\), then
    \[ A^m = P \begin{bmatrix} \lambda_1^m & & \\ & \lambda_2^m & \\ & & \ddots \\ & & & \lambda_n^m \end{bmatrix} P^{-1} \text{ for } P = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix} \]
For a vector $\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + ... + c_n \mathbf{v}_n$,

$A^n \mathbf{v} = c_1 \lambda_1^n \mathbf{v}_1 + c_2 \lambda_2^n \mathbf{v}_2 + ... + c_n \lambda_n^n \mathbf{v}_n$

- If a linear endomorphism $T : V \to V$ on a vector space $V$ of dimension $n$ has $n$ linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, then

$[T^m]_{D \to D} = \begin{bmatrix} \lambda_1^m \\
\lambda_2^m \\
... \\
\lambda_n^m \end{bmatrix}$ for $D = \{ \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \}$

For a vector $\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + ... + c_n \mathbf{v}_n$,

$T^m(\mathbf{v}) = c_1 \lambda_1^m \mathbf{v}_1 + c_2 \lambda_2^m \mathbf{v}_2 + ... + c_n \lambda_n^m \mathbf{v}_n$

- Solve linear recurrence:
  - If it is non-homogeneous, convert it to homogeneous.
  - Find the characteristic polynomial of the recurrence and find its roots $\lambda_1, \lambda_2, ..., \lambda_m$.
  - Use the first $m$ terms of the recurrence to determine $c_1, c_2, ..., c_m$ in $a_n = c_1 \lambda_1^n + c_2 \lambda_2^n + ... + c_m \lambda_m^n$.
- Solve the system $\mathbf{x}' = A\mathbf{x}$ of homogeneous linear ODEs with constant coefficients: If an $n \times n$ matrix $A$ has $n$ linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, then the general solution of $\mathbf{x}' = A\mathbf{x}$ is

$\mathbf{x} = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 + ... + c_n e^{\lambda_n t} \mathbf{v}_n$.

Use the initial condition to determine $c_1, c_2, ..., c_n$.
- Find the orthogonal complement of a subspace $W \subset \mathbb{R}^n$:
  
  $\text{Row}(A) \perp = \text{Nul}(A)$

  $(W^\perp)^\perp = W$

  $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

- Find $\text{proj}_W \mathbf{v}$ and $\text{proj}_{W^\perp} \mathbf{v}$.
- Gram-Schmidt algorithm.
- Orthogonal Diagonalization and Spectral Decomposition: For an $n \times n$ real symmetric matrix $A$,
  - find $n$ linearly independent eigenvectors $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n$ of $A$,
  - apply Gram-Schmidt to $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n$ to obtain an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ and
\[ Q^T A Q = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix} \text{ for } Q = [v_1 \ v_2 \ \ldots \ v_n] \]

and
\[ A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \ldots + \lambda_n v_n v_n^T \]

where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues associated to \( v_1, v_2, \ldots, v_n \).

Review problems:

(1) For each of the following maps \( T \),
- Determine whether \( T \) is a linear transformation.
- If \( T \) is a linear transformation, determine whether \( T \) is 1-1, onto and/or bijective.

You must justify your answer:
(a) \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) given by \( T(x, y, z) = (x - y, y - z, z - x) \).
(b) \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) given by \( T(x, y, z) = (x + y + z, x + 2y + 3z) \).
(c) \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) given by \( T(f(x)) = f(x^2) + x \).
(d) \( T : M_{3 \times 3}(\mathbb{R}) \to M_{3 \times 3}(\mathbb{R}) \) given by \( T(A) = A + A^T \).

(2) Let \( M_{m \times n}(\mathbb{R}) \) be the vector space of \( m \times n \) real matrices and \( T : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R}) \) be the map given by
\[ T(A) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} A \]
for all \( A \in M_{2 \times 2}(\mathbb{R}) \).
(a) Show that \( T \) is a linear transformation.
(b) Find the kernel, range and rank of \( T \).
(c) Is \( T \) onto, 1-1 and/or bijective? Justify your answer.
(d) Find the characteristic polynomial, eigenvalues and eigenvectors of \( T \).
(e) Is \( T \) diagonalizable? If it is, find a basis \( B \) of \( M_{2 \times 2}(\mathbb{R}) \) such that \( [T]_{B \to B} \) is diagonal.
(f) Compute
\[ T^{2016} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

(3) Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear endormorphism given by
\[ T(x, y, z) = (2x + y + z, x + 2y + z, x + y + 2z) \]
(a) Find the kernel, range and rank of \( T \).
(b) Is $T$ onto, 1-1 and/or bijective?
(c) Find the matrix $[T]_{B\leftarrow B}$ representing $T$ under the standard basis $B$.
(d) Find the matrix $[T]_{C\leftarrow C}$ representing $T$ under the basis

$$C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$ 

(e) Find the eigenvalues, eigenvectors and characteristic polynomial of $T$.
(f) Is $T$ diagonalizable? If it is, find a basis $D$ such that $[T]_{D\leftarrow D}$ is diagonal.

(4) Let $P_3$ be the vector space of real polynomials of degree $\leq 3$ and $T : P_3 \rightarrow P_3$ be the map given by

$$T(f(x)) = xf'(x) + f(1).$$

(a) Show that $T$ is a linear endomorphism.
(b) Find the kernel, range and rank of $T$.
(c) Find the characteristic polynomial, eigenvalues and eigenvectors of $T$.
(d) Is $T$ diagonalizable? If it is, find a basis $B$ of $P_3$ such that $[T]_{B\leftarrow B}$ is diagonal.

(5) Let $B$ and $C$ be two $n \times n$ invertible matrices. Show that all three matrices $CBC, BC^2$ and $C^2B$ have the same characteristic polynomials.

(6) Let $W$ be the subspace of $\mathbb{R}^4$ given by

$$W = \{ (x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 = x_2 + x_3 + x_4 = 0 \}.$$ 

(a) Find an orthonormal basis for $W$.
(b) Find the projection of $v = (1, 1, 1, 1)$ onto $W$.
(c) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear transformation given by

$$T(v) = \text{proj}_W v.$$ 

Find the kernel and range of $T$ and find $[T]_{B\leftarrow B}$ under the standard basis $B$ of $\mathbb{R}^4$.

(7) Let $A$ be a real symmetric matrix whose characteristic polynomial is $x^4 - 2x^2 + 1$.

(a) Show that $A$ is invertible.
(b) Find the characteristic polynomial of $A + A^{-1}$.
(8) Let $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ be two linear transformations between two vector spaces $V$ and $W$. Show that
\[ K(T_1) \cap K(T_1 - T_2) = K(T_1 + T_2) \cap K(T_2). \]

(9) Find all $n \times n$ real matrices $A$ with the property that every nonzero vector in $\mathbb{R}^n$ is an eigenvector of $A$. Justify your answer.

(10) Let $\{a_n : n = 0, 1, 2, \ldots \}$ be a sequence satisfying
\[ a_{n+2} = 5a_{n+1} - 6a_n + 1 \]
for all $n \geq 0$ and $a_0 = a_1 = 1$. Find a formula for $a_n$.

(11) Let $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be two linear transformations satisfying $T_1 \circ T_2 = T_2 \circ T_1 = 0$,
\[ T_1(1,1) = (2,1) \text{ and } T_2(1,2) = (1,0). \]
(a) Find $T_1$ and $T_2$.
(b) Find the kernels and ranges of $T_1$ and $T_2$.

(12) Construct the following examples (You must justify your examples):
(a) Three vectors $v_1, v_2, v_3$ in $\mathbb{R}^3$ such that each pair of the three are linearly independent but $v_1, v_2, v_3$ are linearly dependent.
(b) Two linear transformations $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $\text{rank}(T_1 \circ T_2) = 1$ and $\text{rank}(T_2 \circ T_1) = 0$.
(c) Two $3 \times 3$ matrices with the same characteristic polynomials but not similar.
(d) A linear endomorphism $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that the characteristic polynomial of $T$ is $x^3$ and $T^2 \neq 0$.
(e) Two $3 \times 3$ matrices $A$ and $B$ such that both $A$ and $B$ are diagonalizable but $A + B$ is not diagonalizable.
(f) Two $3 \times 3$ matrices $A$ and $B$ such that both $A$ and $B$ are diagonalizable but $AB$ is not diagonalizable.
(g) Four $2 \times 2$ matrices $A_1, A_2, B_1, B_2$ such that $A_1 \sim B_1$ and $A_2 \sim B_2$ but $A_1 + A_2 \not\sim B_1 + B_2$.
(h) Four $2 \times 2$ matrices $A_1, A_2, B_1, B_2$ such that $A_1 \sim B_1$ and $A_2 \sim B_2$ but $A_1 A_2 \not\sim B_1 B_2$.

(13) Show that two real symmetric matrices with the same characteristic polynomials must be similar.

(14) Determine whether the following matrices $A$ and $B$ are similar. Justify your answers.
(a) \( A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \)

(c) \( A = \begin{bmatrix} 1 & 2016 & 2014 \\ 2 & 2015 & 3 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 & 2015 & 2016 \\ 3 & 2014 & 1 \end{bmatrix} \)

(15) Orthogonally diagonalize and find spectral decompositions of the following symmetric matrices:

a) \( \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \) b) \( \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \)

(16) Solve the following ODEs:

a) \[
\begin{align*}
\frac{dx_1}{dt} &= 2x_1 + x_2 + x_3 \\
\frac{dx_2}{dt} &= x_1 + 2x_2 + x_3 \\
\frac{dx_3}{dt} &= x_1 + x_2 + 2x_3 \\
\end{align*}
\]

with \( x_1(1) = 1 \), \( x_2(1) = 2 \), \( x_3(1) = 3 \)

b) \( \frac{dx^2}{dt^2} - 5 \frac{dx}{dt} + 6x = 0 \) with \( x(1) = 2 \).

(17) Let \( A \) be an \( n \times n \) symmetric matrix with characteristic polynomial \( (x - 1)^{n-1}(x + 1) \) and

\[
\text{Nul}(A + I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\}.
\]

Find \( A \).