Linear Algebra II Lecture 11

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Outline

1. Eigenvalues, Eigenvectors, Characteristic Polynomials of Matrices

2. Diagonalization
Given an $n \times n$ matrix $A$,

- $\det(xI - A)$ is the characteristic polynomial of $A$. It is a polynomial of degree $n$ and leading coefficient 1:

$$\det(xI - A) = x^n + a_1x^{n-1} + \ldots + a_n$$

where $a_n = \det(-A) = (-1)^n \det(A)$ and $a_1 = -\text{Tr}(A)$.

- The roots of $\det(xI - A)$ are the eigenvalues of $A$.

- For each eigenvalue $\lambda$, the space $\text{Nul}(\lambda I - A) \subset \mathbb{R}^n$ the eigenspace of $A$ corresponding to $\lambda$ and every nonzero vector $v \in \text{Nul}(\lambda I - A)$ is an eigenvector of $A$ corresponding to $\lambda$:

$$v \in \text{Nul}(\lambda I - A) \iff (\lambda I - A)v = 0 \iff Av = \lambda v.$$
Given an $n \times n$ matrix $A$,

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v \in \text{Nul}(\lambda I - A) \iff (\lambda I - A)v = 0 \iff Av = \lambda v.
$$
An alternative convention of CP is \( \det(A - xI) \):

\[
\det(A - xI) = (-1)^n \det(xI - A)
\]

\[
\det(A - \lambda I) = 0 \iff \det(\lambda I - A) = 0
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\text{Nul}(A - \lambda I) = \text{Nul}(\lambda I - A)
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Eigenvalues, eigenspaces and eigenvectors remain the same.
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Eigenvalues, eigenspaces and eigenvectors remain the same.
Example

Find the eigenvalues, eigenvectors and characteristic polynomial of

\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \].

Computation of CP:

\[ \det(xI - A) = \det \begin{bmatrix} x - 1 & -1 & -1 \\ -1 & x - 1 & -1 \\ -1 & -1 & x - 1 \end{bmatrix} \]

\[ = (x - 1) \det \begin{bmatrix} x - 1 & -1 \\ -1 & x - 1 \end{bmatrix} - (-1) \det \begin{bmatrix} -1 & -1 \\ -1 & x - 1 \end{bmatrix} + (-1) \det \begin{bmatrix} -1 & -1 \\ x - 1 & -1 \end{bmatrix} = x^3 - 3x^2 = x^2(x - 3) \]
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\]

\[
+ (-1) \det \begin{bmatrix}
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Example

It has two eigenvalues 0 and 3 with eigenspaces:

\[ \text{Nul}(A - 0I) = \text{Nul} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \]

\[ \text{Nul}(A - 3I) = \text{Nul} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \]
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Similar Matrices

Two $n \times n$ matrices $A$ and $B$ are similar if there exists an invertible $n \times n$ matrix $P$ such that $B = P^{-1}AP$, written as $A \sim B$.

If $A \sim B$, then

- $\det(A) = \det(B)$ and $\operatorname{Tr}(A) = \operatorname{Tr}(B)$;
- $A$ and $B$ have the same characteristic polynomials, i.e.,
  \[
  \det(xI - A) = \det(xI - B)
  \]
- $A$ and $B$ have the same eigenvalues;
- the eigenspaces of $A$ and $B$ have the same dimensions, i.e.,
  \[
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  for all $\lambda$.  

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for all $\lambda$. 
An $n \times n$ matrix $A$ is **diagonalizable** if $A$ is similar to a diagonal matrix, i.e., there exists an invertible matrix $P$ such that

$$P^{-1}AP = \begin{bmatrix} 
\lambda_1 & & \\
& \lambda_2 & \\
& & \vdots \\
& & & \lambda_n
\end{bmatrix}. $$

If the above holds,

$$\det(xI - A) = (x - \lambda_1)(x - \lambda_2)\ldots(x - \lambda_n)$$

and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$. 
An $n \times n$ matrix $A$ is **diagonalizable** if $A$ is similar to a diagonal matrix, i.e., there exists an invertible matrix $P$ such that

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and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$. 
Criterion of Diagonalization

Theorem

An \( n \times n \) matrix \( A \) is diagonalizable if and only if one of the following holds:

- \( A \) has \( n \) linearly independent eigenvectors.
- The sum of the dimensions of the eigenspaces of \( A \) is \( n \):
  \[
  \sum_{\lambda} \dim \text{Nul}(\lambda I - A) = n.
  \]

The eigenvectors associated to different eigenvalues are always linear independent. So \( A \) has \( n \) linearly independent eigenvectors if and only if the sum of the dimensions of the eigenspaces of \( A \) is \( n \).
An $n \times n$ matrix $A$ is diagonalizable if and only if one of the following holds:

- $A$ has $n$ linearly independent eigenvectors.
- The sum of the dimensions of the eigenspaces of $A$ is $n$:
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The eigenvectors associated to different eigenvalues are always linear independent. So $A$ has $n$ linearly independent eigenvectors if and only if the sum of the dimensions of the eigenspaces of $A$ is $n$. 
Proof. If $A$ is diagonalizable, there is an invertible $P$ such that

$$P^{-1}AP = \begin{bmatrix}
\lambda_1 & & \\
& \lambda_2 & \\
& & \ddots \\
& & & \lambda_n
\end{bmatrix} \Rightarrow AP = P \begin{bmatrix}
\lambda_1 & & \\
& \lambda_2 & \\
& & \ddots \\
& & & \lambda_n
\end{bmatrix}$$

Let $P = [v_1 \ v_2 \ \ldots \ \ v_n]$. Then

$$AP = [Av_1 \ Av_2 \ \ldots \ \ Av_n] = [\lambda_1 v_1 \ \lambda_2 v_2 \ \ldots \ \lambda_n v_n] = PD.$$ 

Therefore, $Av_i = \lambda_i v_i$. And since $P$ is invertible, $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors of $A$. 
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Let $P = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$. Then

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Let $P = [v_1 \ v_2 \ \cdots \ \ v_n]$. Then

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Therefore, $Av_i = \lambda_i v_i$. And since $P$ is invertible, $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors of $A$. 

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Proof. If $A$ has $n$ linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, we let $P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_n]$. Then

$$AP = [A\mathbf{v}_1 \ A\mathbf{v}_2 \ \ldots \ A\mathbf{v}_n] = [\lambda_1 \mathbf{v}_1 \ \lambda_2 \mathbf{v}_2 \ \ldots \ \lambda_n \mathbf{v}_n] = PD.$$ 

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, $P$ is invertible. So

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \vdots \\ & & & \lambda_n \end{bmatrix}.$$
Criterion of Diagonalization

Proof. If $A$ has $n$ linearly independent eigenvectors $v_1, v_2, \ldots, v_n$, we let $P = [v_1 \ v_2 \ \ldots \ v_n]$. Then

$$AP = [Av_1 \ Av_2 \ \ldots \ Av_n] = [\lambda_1 v_1 \ \lambda_2 v_2 \ \ldots \ \lambda_n v_n] = PD.$$ 

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$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 \\ & \lambda_2 \\ & & \ldots \\ & & & \lambda_n \end{bmatrix}.$$
Corollary

An $n \times n$ matrix $A$ with $n$ distinct eigenvalues is diagonalizable.

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of $A$ and $v_1, v_2, ..., v_n$ be $n$ eigenvectors of $A$ associated to $\lambda_1, \lambda_2, ..., \lambda_n$. Since eigenvectors associated to different eigenvalues are linearly independent, $v_1, v_2, ..., v_n$ are linearly independent. So $A$ is diagonalizable.

Theorem

Every real symmetric matrix is diagonalizable.
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Diagonalizable Matrices

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Theorem

Every real symmetric matrix is diagonalizable.
Example

Is

\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]
diagonalizable? If it is, find invertible \( P \) such that \( P^{-1}AP \) is diagonal.

Solution. Since \( \dim \text{Nul}(A - 0I) + \dim \text{Nul}(A - 3I) = 3 \), \( A \) is diagonalizable.

It has 3 linearly independent eigenvectors \( \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). So we let \( P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \) and \( P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \).
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Example

Is \( A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \) diagonalizable? If it is, find invertible \( P \) such that \( P^{-1}AP \) is diagonal.

Solution. The characteristic polynomial of \( A \) is

\[
\det(xI - A) = \det \begin{bmatrix} x - 1 & -1 \\ 1 & x + 1 \end{bmatrix} = (x - 1)(x + 1) + 1 = x^2.
\]

So it has one eigenvalue 0 with eigenspace

\[
\text{Nul}(A - 0I) = \text{Nul} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.
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Since \( \dim \text{Nul}(A - 0I) = 1 < 2 \), \( A \) is not diagonalizable.
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Since \( \dim \text{Nul}(A - 0I) = 1 < 2 \), \( A \) is not diagonalizable.
Example

Find $x$, $y$, $z$ such that $A = \begin{pmatrix} 2 & x & y \\ 2 & z \\ 2 \end{pmatrix}$ is diagonalizable.

Solution. $A$ has only one eigenvalue 2. So it is diagonalizable if and only if

$$\dim \text{Nul}(A - 2I) = 3.$$ 

By Rank Theorem,

$$\dim \text{Nul}(A - 2I) + \text{rank}(A - 2I) = 3.$$ 

So $\dim \text{Nul}(A - 2I) = 3$ if and only if $\text{rank}(A - 2I) = 0$. Therefore, $A$ is diagonalizable if and only if $\text{rank}(A - 2I) = 0$, i.e., $A - 2I = 0$. So $A = 2I$ and $x = y = z = 0$. 
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By Rank Theorem,

$$\dim \text{Nul}(A - 2I) + \text{rank}(A - 2I) = 3.$$ 

So $\dim \text{Nul}(A - 2I) = 3$ if and only if $\text{rank}(A - 2I) = 0$. Therefore, $A$ is diagonalizable if and only if $\text{rank}(A - 2I) = 0$, i.e., $A - 2I = 0$. So $A = 2I$ and $x = y = z = 0$. 

Xi Chen
Linear Algebra II Lecture 11
Example

Find $x, y, z$ such that $A = \begin{bmatrix} 2 & x & y \\ 2 & z & \end{bmatrix}$ is diagonalizable.

Solution. Suppose that $P^{-1}AP = D$ is diagonal for some invertible $P$.
Since the characteristic polynomial of $A$ is $(x - 2)^3$, $D$ must be

$$D = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2I.$$ 

Therefore,

$$P^{-1}AP = D \Rightarrow A = PDP^{-1} = P(2I)P^{-1} = 2PIP^{-1} = 2I.$$ 

So $x = y = z = 0$. 
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Suppose that $P^{-1}AP = D$. Then

$$(P^{-1}AP)^n = D^n \Rightarrow P^{-1}A^nP = D^n \Rightarrow A^n = P^{-1}D^nP.$$ 

If $D$ is diagonal, then

$$A^n = P^{-1} \begin{bmatrix} 
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1 & \lambda_2 & \ldots & \lambda_n 
\end{bmatrix}^n P = P^{-1} \begin{bmatrix} 
\lambda_1^n & \lambda_2^n & \ldots & \lambda_n^n \\
\lambda_1^n & \lambda_2^n & \ldots & \lambda_n^n \\
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\lambda_1^n & \lambda_2^n & \ldots & \lambda_n^n 
\end{bmatrix} P$$
Example

Find a formula for $A^n$, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Solution. $A$ has eigenvalues 1 and 3 with eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So $P^{-1}AP = D$ for

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Then

$$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 3^n + 1 & 3^n - 1 \\ 3^n - 1 & 3^n + 1 \end{bmatrix}.$$
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