Outline

1. Injection, Surjection and Isomorphism

2. Rank Theorem
Definition of Injection, Surjection and Bijection

We call a map $f : X \rightarrow Y$ an injection (injective, one-to-one, 1-1) if $f(x_1) \neq f(x_2)$ for all $x_1 \neq x_2 \in X$.

We call a map $f : X \rightarrow Y$ a surjection (surjective, onto) if $f(X) = Y$, i.e., for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

We call a map $f : X \rightarrow Y$ a bijection (bijective) if it is 1-1 and onto. A bijection $f : X \rightarrow Y$ has an inverse $f^{-1} : Y \rightarrow X$ such that $f \circ f^{-1} = 1_Y$ and $f^{-1} \circ f = 1_X$, where $1_X$ and $1_Y$ are the identity maps on $X$ and $Y$. A map $f$ is a bijection if and only if $f^{-1}$ exists.

For example, $f(x) = x^3 : \mathbb{R} \rightarrow \mathbb{R}$ is 1-1 and onto and hence bijective; $f(x) = x^2 : \mathbb{R} \rightarrow [0, \infty)$ is onto and not 1-1.
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Isomorphism

A linear transformation $T : V \to W$ is an isomorphism if it is an bijection. Two vector spaces are isomorphic (written as $V \cong W$) if there is an isomorphism $T : V \to W$.

- Let $T : \mathbb{R} \to \mathbb{R}$ be given by $T(x) = 2x$. Then $T$ is an isomorphism since $T$ is 1-1 and onto.
- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (x + y, x - y)$. Then $T$ is an isomorphism since $T^{-1}$ exists: $T^{-1}(x, y) = (\frac{1}{2}(x + y), \frac{1}{2}(x - y))$.
- Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be given by $T(f(x)) = f(x + 1)$. Then $T$ is an isomorphism since $T^{-1}$ exists: $T^{-1}(f(x)) = f(x - 1)$. 

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Linear Algebra II Lecture 9
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Let $T : V \rightarrow W$ be a linear transformation. Then

- $T$ is 1-1 if and only if $K(T) = \{0\}$, i.e., $\dim K(T) = 0$;
- $T$ is onto if and only if $R(T) = W$, i.e., $\text{rank}(T) = \dim W$ if $\dim W < \infty$.

Proof. If $T$ is 1-1, then $T(v) \neq T(0) = 0$ for all $v \neq 0$. Therefore, $v \notin K(T)$ for all $v \neq 0$. That is, $K(T) = \{0\}$.
Suppose that $K(T) = \{0\}$. If $T$ is not 1-1, then there exists $v_1 \neq v_2$ such that $T(v_1) = T(v_2)$. Then

$$T(v_1 - v_2) = T(v_1) - T(v_2) = 0$$

$\Rightarrow 0 \neq v_1 - v_2 \in K(T)$, Contradiction.
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Inverse Linear Transformation

**Theorem**

Let $T : V \rightarrow W$ be an isomorphism. Then $T^{-1}$ is also a linear transformation.

In addition, if $V$ is finite dimensional, $\dim V = \dim W$ and

$$[T]_{C \leftarrow B}^{-1} = [T^{-1}]_{B \leftarrow C}.$$

**$T^{-1}$ is a linear transformation.**

For $w_1, w_2 \in W$, let $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$. Then

$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = w_1 + cw_2$$

$$\Rightarrow \begin{align*}
T^{-1}(w_1) + cT^{-1}(w_2) &= T^{-1}(w_1 + cw_2) \\
\end{align*}$$
Inverse Linear Transformation

**Theorem**

Let \( T : V \rightarrow W \) be an isomorphism. Then \( T^{-1} \) is also a linear transformation. In addition, if \( V \) is finite dimensional, \( \dim V = \dim W \) and

\[
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Let $T : V \to W$ be an isomorphism. Then $T^{-1}$ is also a linear transformation. In addition, if $V$ is finite dimensional, $\dim V = \dim W$ and

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Proof of dim $V = \text{dim } W$.

If $\text{dim } V = n < \infty$, let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of $V$. Then

$$R(T) = \text{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\}.$$ 

Since $T$ is onto, $R(T) = W$ and hence $\text{dim } W = m \leq n$. Let $C = \{w_1, w_2, \ldots, w_m\}$ be a basis of $W$. Then

$$R(T^{-1}) = \text{Span}\{T^{-1}(w_1), T^{-1}(w_2), \ldots, T^{-1}(w_m)\}.$$ 

Since $T^{-1}$ is onto, $R(T^{-1}) = V$ and hence $\text{dim } V = n \leq m$. So $m = n$. Finally,

$$T \circ T^{-1} = 1_W \Rightarrow [T]_{C \leftarrow B}[T^{-1}]_{B \leftarrow C} = [1_W]_{C \leftarrow C} = I, \text{ i.e., } [T]^{-1}_{C \leftarrow B} = [T^{-1}]_{B \leftarrow C}.$$
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Definition

Let \( T : V \rightarrow W \) be a linear transformation. The rank of \( T \) is the dimension of its range \( R(T) \), i.e.,

\[
\text{rank}(T) = \dim R(T).
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Theorem

Let \( T : V \rightarrow W \) be a linear transformation between two vector spaces of finite dimensions. Then

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Let $T: V \rightarrow W$ be a linear transformation. If $\dim V < \infty$, then

$$\dim K(T) + \operatorname{rank}(T) = \dim V.$$ 

Here are some remarks:

- $\operatorname{rank}(T) \leq \dim V$; and since $R(T) \subseteq W$, $\operatorname{rank}(T) \leq \dim W$; therefore,

  $$\operatorname{rank}(T) \leq \min(\dim V, \dim W).$$

For example, a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^5$ has rank at most 3.
Let \( T : V \rightarrow W \) be a linear transformation. If \( \dim V < \infty \), then

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For example, a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ has rank at most 3.
Rank Theorem

- $T$ cannot be onto if $\dim V < \dim W$ since
  \[ \dim R(T) = \text{rank}(T) \leq \dim V < \dim W \Rightarrow R(T) \neq W. \]

- $T$ cannot be 1-1 if $\dim V > \dim W$ since
  \[ \dim K(T) = \dim V - \text{rank}(T) \geq \dim V - \dim W > 0 \]
  \[ \Rightarrow K(T) \neq \{0\}. \]

- $\dim R(T) = \dim V$ if and only if $K(T) = \{0\}$, i.e., $T$ is 1-1.
- If $\dim V = \dim W$, $T$ is 1-1 if and only if $T$ is onto since
  \[ T \text{ is 1-1} \iff K(T) = \{0\} \iff \dim K(T) = 0 \]
  \[ \iff \text{rank}(T) = \dim V = \dim W \iff R(T) = W \]
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  \dim R(T) = \text{rank}(T) \leq \dim V < \dim W \Rightarrow R(T) \neq W.
  \]

- $T$ cannot be 1-1 if $\dim V > \dim W$ since
  \[
  \dim K(T) = \dim V - \text{rank}(T) \geq \dim V - \dim W > 0
  \Rightarrow K(T) \neq \{0\}.
  \]

- $\dim R(T) = \dim V$ if and only if $K(T) = \{0\}$, i.e., $T$ is 1-1.
  - If $\dim V = \dim W$, $T$ is 1-1 if and only if $T$ is onto since
    \[
    T \text{ is 1-1} \iff K(T) = \{0\} \iff \dim K(T) = 0
    \iff \text{rank}(T) = \dim V = \dim W \iff R(T) = W
    \]
Rank Theorem

- $T$ cannot be onto if $\dim V < \dim W$ since
  \[ \dim R(T) = \text{rank}(T) \leq \dim V < \dim W \Rightarrow R(T) \neq W. \]

- $T$ cannot be 1-1 if $\dim V > \dim W$ since
  \[ \dim K(T) = \dim V - \text{rank}(T) \geq \dim V - \dim W > 0 \]
  \[ \Rightarrow K(T) \neq \{0\}. \]

- $\dim R(T) = \dim V$ if and only if $K(T) = \{0\}$, i.e., $T$ is 1-1.
- If $\dim V = \dim W$, $T$ is 1-1 if and only if $T$ is onto since
  \[ T \text{ is 1-1} \iff K(T) = \{0\} \iff \dim K(T) = 0 \]
  \[ \iff \text{rank}(T) = \dim V = \dim W \iff R(T) = W. \]
Proof of Rank Theorem

Let \( \dim K(T) = k \) and \( \dim V = n \). There is a basis \( \{v_1, v_2, \ldots, v_k\} \) for \( K(T) \). Since \( \{v_1, v_2, \ldots, v_k\} \) is linearly independent, we can expand it to a basis of \( V \):

\[
\{v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_n\}.
\]

Since

\[
R(T) = \text{Span}\{T(v_1), T(v_2), \ldots, T(v_k), T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}
\]

\[
= \text{Span}\{0, 0, \ldots, 0, T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}
\]

\[
= \text{Span}\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}
\]

It suffices to show that \( \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\} \) is linearly independent.
Proof of Rank Theorem

Let $\dim K(T) = k$ and $\dim V = n$. There is a basis $\{v_1, v_2, ..., v_k\}$ for $K(T)$. Since $\{v_1, v_2, ..., v_k\}$ is linearly independent, we can expand it to a basis of $V$:

$$\{v_1, v_2, ..., v_k, v_{k+1}, v_{k+2}, ..., v_n\}.$$  

Since

$$R(T) = \text{Span}\{T(v_1), T(v_2), ..., T(v_k), T(v_{k+1}), T(v_{k+2}), ..., T(v_n)\}$$ 

$$= \text{Span}\{0, 0, ..., 0, T(v_{k+1}), T(v_{k+2}), ..., T(v_n)\}$$ 

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It suffices to show that $\{T(v_{k+1}), T(v_{k+2}), ..., T(v_n)\}$ is linearly independent.
Proof of Rank Theorem

Let \( \dim K(T) = k \) and \( \dim V = n \). There is a basis \( \{v_1, v_2, \ldots, v_k\} \) for \( K(T) \). Since \( \{v_1, v_2, \ldots, v_k\} \) is linearly independent, we can expand it to a basis of \( V \):

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Since

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\]

It suffices to show that \( \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\} \) is linearly independent.
Linear Independence of \( \{ T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n) \} \)

If \( T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n) \) are linearly dependent, then there exist \( a_{k+1}, a_{k+2}, \ldots, a_n \), not all zero, such that

\[
a_{k+1} T(v_{k+1}) + a_{k+2} T(v_{k+2}) + \ldots + a_n T(v_n) = 0.
\]

Then

\[
T(a_{k+1}v_{k+1} + a_{k+2}v_{k+2} + \ldots + a_n v_n) = 0 \Rightarrow
\]

\[
a_{k+1}v_{k+1} + a_{k+2}v_{k+2} + \ldots + a_n v_n \in K(T) \Rightarrow
\]

\[
a_{k+1}v_{k+1} + a_{k+2}v_{k+2} + \ldots + a_n v_n = a_1v_1 + a_2v_2 + \ldots + a_k v_k \Rightarrow
\]

\[
-a_1v_1 - a_2v_2 - \ldots - a_k v_k + a_{k+1}v_{k+1} + a_{k+2}v_{k+2} + \ldots + a_n v_n = 0.
\]

Therefore, \( v_1, v_2, \ldots, v_n \) are linearly dependent as

\( a_{k+1}, a_{k+2}, \ldots, a_n \) are not all zero. Contradiction.
Linear Independence of \( \{ T(v_{k+1}), T(v_{k+2}), ..., T(v_n) \} \)

If \( T(v_{k+1}), T(v_{k+2}), ..., T(v_n) \) are linearly dependent, then there exist \( a_{k+1}, a_{k+2}, ..., a_n \), not all zero, such that

\[
a_{k+1} T(v_{k+1}) + a_{k+2} T(v_{k+2}) + ... + a_n T(v_n) = 0.
\]

Then

\[
T(a_{k+1} v_{k+1} + a_{k+2} v_{k+2} + ... + a_n v_n) = 0 \Rightarrow \\
a_{k+1} v_{k+1} + a_{k+2} v_{k+2} + ... + a_n v_n \in K(T) \Rightarrow \\
a_{k+1} v_{k+1} + a_{k+2} v_{k+2} + ... + a_n v_n = a_1 v_1 + a_2 v_2 + ... + a_k v_k \Rightarrow \\
- a_1 v_1 - a_2 v_2 - ... - a_k v_k + a_{k+1} v_{k+1} + a_{k+2} v_{k+2} + ... + a_n v_n = 0.
\]

Therefore, \( v_1, v_2, ..., v_n \) are linearly dependent as \( a_{k+1}, a_{k+2}, ..., a_n \) are not all zero. Contradiction.