Linear Algebra II Lecture 8

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Outline

1. Matrix Representation of Linear Transformations
2. Spaces of Linear Transformations
3. Kernel and Range
Matrix Representations of $T_1 + T_2$ and $T_1 \circ T_2$

Let $T_1 : V \to W$ and $T_2 : V \to W$ be two linear transformations from $V \to W$ and let $B$ and $C$ be two ordered bases of $V$ and $W$, respectively. Then for all $c \in \mathbb{R}$,

$$[T_1 + T_2]_C \leftarrow B = [T_1]_C \leftarrow B + [T_2]_C \leftarrow B \quad \text{and} \quad [cT_1]_C \leftarrow B = c[T_1]_C \leftarrow B$$

Let $T_1 : V \to W$ and $T_2 : U \to V$ be two linear transformations and let $B$, $C$, $D$ be three ordered bases of $U$, $V$, $W$, respectively. Then

$$[T_1 \circ T_2]_D \leftarrow B = [T_1]_D \leftarrow C[T_2]_C \leftarrow B$$
Let $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ be two linear transformations from $V \rightarrow W$ and let $B$ and $C$ be two ordered bases of $V$ and $W$, respectively. Then for all $c \in \mathbb{R}$,

$$\begin{align*}
[T_1 + T_2]_{C \leftarrow B} &= [T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B} \\
[cT_1]_{C \leftarrow B} &= c[T_1]_{C \leftarrow B}
\end{align*}$$

Let $T_1 : V \rightarrow W$ and $T_2 : U \rightarrow V$ be two linear transformations and let $B, C, D$ be three ordered bases of $U, V, W$, respectively. Then

$$[T_1 \circ T_2]_{D \leftarrow B} = [T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B}$$
Let $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ be two linear transformations from $V \rightarrow W$ and let $B$ and $C$ be two ordered bases of $V$ and $W$, respectively. Then for all $c \in \mathbb{R}$,

$$[T_1 + T_2]_{C \leftarrow B} = [T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B} \text{ and}$$

$$[cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}$$

Let $T_1 : V \rightarrow W$ and $T_2 : U \rightarrow V$ be two linear transformations and let $B$, $C$, $D$ be three ordered bases of $U$, $V$, $W$, respectively. Then

$$[T_1 \circ T_2]_{D \leftarrow B} = [T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B}$$
Proofs for \([T_1 + T_2]\) and \([cT_1]\)

For \(T_1 : V \rightarrow W, T_2 : V \rightarrow W\) and \(v \in V\),

\[
[(T_1 + T_2)(v)]_C = [T_1 + T_2]_{C \leftarrow B}[v]_B
\]

On the other hand,

\[
[(T_1 + T_2)(v)]_C = [T_1(v) + T_2(v)]_C = [T_1(v)]_C + [T_2(v)]_C
\]

\[
= [T_1]_{C \leftarrow B}[v]_B + [T_2]_{C \leftarrow B}[v]_B
\]

\[
= ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})[v]_B
\]

So \([T_1 + T_2]_{C \leftarrow B} = ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B}).\)

Similarly, Since

\[
[cT_1(v)]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}[v]_B
\]

\[
[cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}.
\]
Proofs for \([T_1 + T_2]\) and \([cT_1]\)

For \(T_1 : V \to W, T_2 : V \to W\) and \(v \in V\),

\[\left[(T_1 + T_2)(v)\right]_C = [T_1 + T_2]_{C \leftarrow B}[v]_B\]

On the other hand,

\[\left[(T_1 + T_2)(v)\right]_C = [T_1(v) + T_2(v)]_C = [T_1(v)]_C + [T_2(v)]_C = [T_1]_{C \leftarrow B}[v]_B + [T_2]_{C \leftarrow B}[v]_B = ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})[v]_B\]

So \([T_1 + T_2]_{C \leftarrow B} = ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})\).

Similarly, Since

\[\left[cT_1(v)\right]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}[v]_B\]

\[\left[cT_1\right]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}\].
Proofs for \([T_1 + T_2] \) and \([cT_1] \)

For \(T_1 : V \rightarrow W \), \(T_2 : V \rightarrow W \) and \(v \in V\),

\[
[(T_1 + T_2)(v)]_C = [T_1 + T_2]_{C \leftarrow B}[v]_B
\]

On the other hand,

\[
[(T_1 + T_2)(v)]_C = [T_1(v) + T_2(v)]_C = [T_1(v)]_C + [T_2(v)]_C \\
= [T_1]_{C \leftarrow B}[v]_B + [T_2]_{C \leftarrow B}[v]_B \\
= ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})[v]_B
\]

So \([T_1 + T_2]_{C \leftarrow B} = ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B}).\)

Similarly, Since

\[
[cT_1(v)]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}[v]_B
\]

\[
[cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}.
\]
Proofs for \([T_1 + T_2]\) and \([cT_1]\)

For \(T_1 : V \rightarrow W, T_2 : V \rightarrow W\) and \(v \in V\),

\[
[(T_1 + T_2)(v)]_C = [T_1 + T_2]_{C \leftarrow B}[v]_B
\]

On the other hand,

\[
[(T_1 + T_2)(v)]_C = [T_1(v) + T_2(v)]_C = [T_1(v)]_C + [T_2(v)]_C
\]

\[
= [T_1]_{C \leftarrow B}[v]_B + [T_2]_{C \leftarrow B}[v]_B
\]

\[
= ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B})[v]_B
\]

So \([T_1 + T_2]_{C \leftarrow B} = ([T_1]_{C \leftarrow B} + [T_2]_{C \leftarrow B}).\)

Similarly, Since

\[
[cT_1(v)]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}[v]_B
\]

\[
[cT_1]_{C \leftarrow B} = c[T_1]_{C \leftarrow B}.
\]
Proofs for $[T_1 \circ T_2]$ 

For $T_1 : V \to W$, $T_2 : U \to V$ and $u \in U$, 

$$[(T_1 \circ T_2)(u)]_D = [T_1 \circ T_2]_{D \leftarrow B}[u]_B$$

On the other hand, 

$$[(T_1 \circ T_2)(u)]_D = [T_1(T_2(u))]_D = [T_1]_{D \leftarrow C}[T_2(u)]_C$$

$$= [T_1]_{D \leftarrow C}[T_2]_C \leftarrow B[u]_B$$

So $[T_1 \circ T_2]_{D \leftarrow B} = ([T_1]_{D \leftarrow C}[T_2]_C \leftarrow B)$. 
Proofs for $[T_1 \circ T_2]$

For $T_1 : V \rightarrow W$, $T_2 : U \rightarrow V$ and $u \in U$,

$$[(T_1 \circ T_2)(u)]_D = [T_1 \circ T_2]_{D\leftarrow B}[u]_B$$

On the other hand,

$$[(T_1 \circ T_2)(u)]_D = [T_1(T_2(u))]_D = [T_1]_{D\leftarrow C}[T_2(u)]_C$$
$$= [T_1]_{D\leftarrow C}[T_2]_C\leftarrow B[u]_B$$

So $[T_1 \circ T_2]_{D\leftarrow B} = ([T_1]_{D\leftarrow C}[T_2]_C\leftarrow B)$. 
Proofs for $[T_1 \circ T_2]$

For $T_1 : V \to W$, $T_2 : U \to V$ and $u \in U$,

$$[(T_1 \circ T_2)(u)]_D = [T_1 \circ T_2]_{D \leftarrow B}[u]_B$$

On the other hand,

$$[(T_1 \circ T_2)(u)]_D = [T_1(T_2(u))]_D = [T_1]_{D \leftarrow C}[T_2(u)]_C$$
$$= [T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B}[u]_B$$

So $[T_1 \circ T_2]_{D \leftarrow B} = ([T_1]_{D \leftarrow C}[T_2]_{C \leftarrow B})$. 
Let $T : V \rightarrow W$ be a linear transformation, let $B$ and $B'$ be two ordered bases of $V$ and let $C$ and $C'$ be two ordered bases of $W$. Then

$$[T]_{C' \leftarrow B'} = P_{C' \leftarrow C}[T]_{C \leftarrow B} P_{B \leftarrow B'}$$

Let $I_V : V \rightarrow V$ and $I_W : W \rightarrow W$ be the identity maps on $V$ and $W$. Then

$$[I_W \circ T \circ I_V]_{C' \leftarrow B'} = [I_W]_{C' \leftarrow C}[T]_{C \leftarrow B}[I_V]_{B \leftarrow B'} = P_{C' \leftarrow C}[T]_{C \leftarrow B} P_{B \leftarrow B'}$$
Let $T : V \to W$ be a linear transformation, let $B$ and $B'$ be two ordered bases of $V$ and let $C$ and $C'$ be two ordered bases of $W$. Then
\[
[T]_{C' \leftarrow B'} = P_{C' \leftarrow C}[T]_{C \leftarrow B}P_{B \leftarrow B'}
\]

Let $I_V : V \to V$ and $I_W : W \to W$ be the identity maps on $V$ and $W$. Then
\[
[I_W \circ T \circ I_V]_{C' \leftarrow B'} = [I_W]_{C' \leftarrow C}[T]_{C \leftarrow B}[I_V]_{B \leftarrow B'}
= P_{C' \leftarrow C}[T]_{C \leftarrow B}P_{B \leftarrow B'}
\]
Vector Space \( L(V, W) \)

**Theorem**

- For all linear transformations \( T_1 : V \rightarrow W \) and \( T_2 : V \rightarrow W \) and \( c \in \mathbb{R} \), \( T_1 + T_2 \) and \( cT_1 \) are also linear transformations from \( V \) to \( W \).

- Furthermore,

\[
[T_1 + T_2]_{B_2 \leftarrow B_1} = [T_1]_{B_2 \leftarrow B_1} + [T_2]_{B_2 \leftarrow B_1}
\]

\[
[cT_1]_{B_2 \leftarrow B_1} = c[T_1]_{B_2 \leftarrow B_1}
\]

where \( B_1 \) is a basis for \( V \) and \( B_2 \) is a basis for \( W \).

- Let \( L(V, W) \) be the set of all linear transformations from \( V \) to \( W \). Then \( L(V, W) \) is itself a vector space over \( \mathbb{R} \) under the addition and scalar multiplication defined above.
Vector Space \( L(V, W) \)

**Theorem**

- For all linear transformations \( T_1 : V \to W \) and \( T_2 : V \to W \) and \( c \in \mathbb{R} \), \( T_1 + T_2 \) and \( cT_1 \) are also linear transformations from \( V \) to \( W \).
- Furthermore,

  \[
  [T_1 + T_2]_{B_2 \leftarrow B_1} = [T_1]_{B_2 \leftarrow B_1} + [T_2]_{B_2 \leftarrow B_1}
  \text{ and }
  [cT_1]_{B_2 \leftarrow B_1} = c[T_1]_{B_2 \leftarrow B_1}
  \]

  where \( B_1 \) is a basis for \( V \) and \( B_2 \) is a basis for \( W \).
- Let \( L(V, W) \) be the set of all linear transformations from \( V \) to \( W \). Then \( L(V, W) \) is itself a vector space over \( \mathbb{R} \) under the addition and scalar multiplication defined above.
Theorem

For all linear transformations $T_1 : V \to W$ and $T_2 : V \to W$ and $c \in \mathbb{R}$, $T_1 + T_2$ and $cT_1$ are also linear transformations from $V$ to $W$.

Furthermore,

$$[T_1 + T_2]_{B_2\leftarrow B_1} = [T_1]_{B_2\leftarrow B_1} + [T_2]_{B_2\leftarrow B_1} \quad \text{and}$$

$$[cT_1]_{B_2\leftarrow B_1} = c[T_1]_{B_2\leftarrow B_1}$$

where $B_1$ is a basis for $V$ and $B_2$ is a basis for $W$.

Let $L(V, W)$ be the set of all linear transformations from $V$ to $W$. Then $L(V, W)$ is itself a vector space over $\mathbb{R}$ under the addition and scalar multiplication defined above.
Let $V$ and $W$ be two vectors spaces over $\mathbb{R}$ of dimensions $\dim V = n$ and $\dim W = m$. Fixing two ordered of bases $B$ of $V$ and $C$ of $W$, there is a map

$$F : L(V, W) \to M_{m \times n}(\mathbb{R})$$

given by $F(T) = [T]_{C \leftarrow B}$

Then $F$ is an invertible linear map. We say $L(V, W) \cong M_{m \times n}(\mathbb{R})$. For example,

$$L(\mathbb{R}^3, \mathbb{R}^2) \cong M_{2 \times 3}(\mathbb{R})$$
$$L(P_4, P_5) \cong M_{6 \times 5}(\mathbb{R})$$
$$L(M_{3 \times 4}(\mathbb{R}), M_{2 \times 3}(\mathbb{R})) \cong M_{6 \times 12}(\mathbb{R})$$
Let $V$ and $W$ be two vector spaces over $\mathbb{R}$ of dimensions $\dim V = n$ and $\dim W = m$. Fixing two ordered bases $B$ of $V$ and $C$ of $W$, there is a map

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For example,

$$L(\mathbb{R}^3, \mathbb{R}^2) \cong M_{2 \times 3}(\mathbb{R})$$

$$L(\mathbb{P}_4, \mathbb{P}_5) \cong M_{6 \times 5}(\mathbb{R})$$

$$L(M_{3 \times 4}(\mathbb{R}), M_{2 \times 3}(\mathbb{R})) \cong M_{6 \times 12}(\mathbb{R})$$
Definition
Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. The \textit{kernel} of $T$ is $K(T) = \ker(T) = \{ x \in V : T(x) = 0 \} \subset V$. The \textit{range} of $T$ is the image of $T$, i.e., $R(T) = T(V) = \{ T(x) : x \in V \} \subset W$.

Theorem
Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. Then $K(T)$ is a subspace of $V$ and $R(T)$ is a subspace of $W$.

Let $T(x, y) = (x, x)$ be a linear transformation from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Then $K(T) = \{(x, y) : T(x, y) = (0, 0)\} = \{(x, y) : x = 0\}$ and $R(T) = \{ T(x, y) \} = \{(x, x)\} = \{(x, y) : x - y = 0\}$.
**Definition**

Let \( T : V \rightarrow W \) be a linear transformation from \( V \) to \( W \). The *kernel* of \( T \) is \( K(T) = \ker(T) = \{ x \in V : T(x) = 0 \} \subset V \). The *range* of \( T \) is the image of \( T \), i.e., \( R(T) = T(V) = \{ T(x) : x \in V \} \subset W \).

**Theorem**

Let \( T : V \rightarrow W \) be a linear transformation from \( V \) to \( W \). Then \( K(T) \) is a subspace of \( V \) and \( R(T) \) is a subspace of \( W \).

Let \( T(x, y) = (x, x) \) be a linear transformation from \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). Then \( K(T) = \{(x, y) : T(x, y) = (0, 0)\} = \{(x, y) : x = 0\} \) and \( R(T) = \{T(x, y)\} = \{(x, x)\} = \{(x, y) : x - y = 0\} \).
**Definition**

Let \( T : V \to W \) be a linear transformation from \( V \) to \( W \). The *kernel* of \( T \) is \( K(T) = \ker(T) = \{ x \in V : T(x) = 0 \} \subset V \). The *range* of \( T \) is the image of \( T \), i.e.,

\[
R(T) = T(V) = \{ T(x) : x \in V \} \subset W.
\]

**Theorem**

Let \( T : V \to W \) be a linear transformation from \( V \) to \( W \). Then \( K(T) \) is a subspace of \( V \) and \( R(T) \) is a subspace of \( W \).

Let \( T(x, y) = (x, x) \) be a linear transformation from \( \mathbb{R}^2 \to \mathbb{R}^2 \). Then \( K(T) = \{ (x, y) : T(x, y) = (0, 0) \} = \{ (x, y) : x = 0 \} \) and \( R(T) = \{ T(x, y) \} = \{ (x, x) \} = \{ (x, y) : x - y = 0 \} \).
Definition

Let $T : V \to W$ be a linear transformation from $V$ to $W$. The \textit{kernel} of $T$ is $K(T) = \ker(T) = \{x \in V : T(x) = 0\} \subset V$. The \textit{range} of $T$ is the image of $T$, i.e., $R(T) = T(V) = \{T(x) : x \in V\} \subset W$.

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Let $T : V \to W$ be a linear transformation from $V$ to $W$. Then $K(T)$ is a subspace of $V$ and $R(T)$ is a subspace of $W$.

Let $T(x, y) = (x, x)$ be a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$. Then $K(T) = \{(x, y) : T(x, y) = (0, 0)\} = \{(x, y) : x = 0\}$ and $R(T) = \{T(x, y)\} = \{(x, x)\} = \{(x, y) : x - y = 0\}$. 
Proof that $K(T)$ and $R(T)$ are subspaces

**$K(T)$ is a subspace of $V$.**
Since $T(0) = 0$, $0 \in K(T)$. For all $v_1, v_2 \in K(T)$, $T(v_1) = T(v_2) = 0$ and hence

$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = 0$$

for all $c \in \mathbb{R}$. Therefore, $v_1 + cv_2 \in K(T)$.

**$R(T)$ is a subspace of $W$.**
Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Thus,

$$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in R(T).$$
Proof that $K(T)$ and $R(T)$ are subspaces

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$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = 0$$

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**$R(T)$ is a subspace of $W$.**

Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Thus,

$$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in R(T)$$.
Proof that $K(T)$ and $R(T)$ are subspaces

**$K(T)$ is a subspace of $V$.**

Since $T(0) = 0$, $0 \in K(T)$. For all $v_1, v_2 \in K(T)$, $T(v_1) = T(v_2) = 0$ and hence

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**$R(T)$ is a subspace of $W$.**

Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Thus,

$$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in R(T).$$
Proof that $K(T)$ and $R(T)$ are subspaces

**$K(T)$ is a subspace of $V$.**

Since $T(0) = 0$, $0 \in K(T)$. For all $v_1, v_2 \in K(T)$, $T(v_1) = T(v_2) = 0$ and hence

$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = 0$$

for all $c \in \mathbb{R}$. Therefore, $v_1 + cv_2 \in K(T)$.

**$R(T)$ is a subspace of $W$.**

Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Thus,

$$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in R(T).$$
If $R(T)$ is finite-dimensional, then the dimension of $R(T)$ is called the \textit{rank} of $T$, denoted by

$$\text{rank}(T) = \dim R(T) = \dim T(V).$$

Given a basis $B = \{v_1, v_2, ..., v_n\}$ of $V$, then the range of a linear transformation $T : V \to W$ is

$$R(T) = T(V) = \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\}.$$ 

Note that

$$\text{rank}(T) = \dim R(T) = \dim \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\} \leq n = \dim V.$$
Range and Rank

If $R(T)$ is finite-dimensional, then the dimension of $R(T)$ is called the *rank* of $T$, denoted by

$$\text{rank}(T) = \dim R(T) = \dim T(V).$$

Given a basis $B = \{v_1, v_2, \ldots, v_n\}$ of $V$, then the range of a linear transformation $T : V \rightarrow W$ is

$$R(T) = T(V) = \text{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\}.$$

Note that

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Range and Rank

If $R(T)$ is finite-dimensional, then the dimension of $R(T)$ is called the rank of $T$, denoted by

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Given a basis $B = \{v_1, v_2, \ldots, v_n\}$ of $V$, then the range of a linear transformation $T : V \to W$ is

$$R(T) = T(V) = \text{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\}.$$

Note that

$$\text{rank}(T) = \dim R(T) = \dim \text{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\} \leq n = \dim V.$$
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by

$$T(v) = Av$$

for an $m \times n$ matrix $A$. Then

$$K(T) = \{v : T(v) = 0\} = \{v : Av = 0\} = \text{Nul}(A).$$

$$R(T) = \text{Span}\{T(e_1), T(e_2), \ldots, T(e_n)\}$$

$$= \text{Span}\{Ae_1, Ae_2, \ldots, Ae_n\} = \text{Col}(A).$$

$$\text{rank}(T) = \dim R(T) = \dim \text{Col}(A) = \text{rank}(A).$$
Kernel, Range and Rank of $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation given by

$$ T(\mathbf{v}) = A\mathbf{v} $$

for an $m \times n$ matrix $A$. Then

$$ K(T) = \{\mathbf{v} : T(\mathbf{v}) = 0\} = \{\mathbf{v} : A\mathbf{v} = 0\} = \text{Nul}(A). $$

$$ R(T) = \text{Span}\{T(\mathbf{e}_1), T(\mathbf{e}_2), ..., T(\mathbf{e}_n)\} = \text{Span}\{A\mathbf{e}_1, A\mathbf{e}_2, ..., A\mathbf{e}_n\} = \text{Col}(A). $$

$$ \text{rank}(T) = \dim R(T) = \dim \text{Col}(A) = \text{rank}(A). $$
Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation given by

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$$K(T) = \{v : T(v) = 0\} = \{v : Av = 0\} = \text{Nul}(A).$$

$$R(T) = \text{Span}\{T(e_1), T(e_2), ..., T(e_n)\}$$

$$= \text{Span}\{Ae_1, Ae_2, ..., Ae_n\} = \text{Col}(A).$$

$$\text{rank}(T) = \dim R(T) = \dim \text{Col}(A) = \text{rank}(A).$$
Kernel, Range and Rank of $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation given by

$$T(v) = Av$$

for an $m \times n$ matrix $A$. Then

$$K(T) = \{v : T(v) = 0\} = \{v : Av = 0\} = \text{Nul}(A).$$

$$R(T) = \text{Span}\{T(e_1), T(e_2), \ldots, T(e_n)\}$$
$$= \text{Span}\{Ae_1, Ae_2, \ldots, Ae_n\} = \text{Col}(A).$$

$$\text{rank}(T) = \dim R(T) = \dim \text{Col}(A) = \text{rank}(A).$$