Outline

1. Linear Dependence
Example. Given \((1, 0), (0, 1)\) and \((1, 1)\) in \(\mathbb{R}^2\), clearly,

\[
\text{Span}\{(1, 0), (0, 1), (1, 1)\} = \text{Span}\{(1, 0), (0, 1)\} = \text{Span}\{(1, 0), (1, 1)\} = \text{Span}\{(0, 1), (1, 1)\}
\]

So one vector among \((1, 0), (0, 1), (1, 1)\) is “redundant”.

**Definition A**

Let \(V\) be a vector space over \(\mathbb{R}\). A nonempty indexed set \(S = \{v_1, v_2, \ldots, v_n, \ldots\}\) of vectors in \(V\) is **linearly independent** if

\[
\text{Span}(S) \neq \text{Span}(S \setminus \{v_i\})
\]

for all \(v_i \in S\). And it is **linearly dependent** if

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for some \(v_i \in S\).
Linear Dependence in Vector Spaces

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for some \(v_i \in S\).
Definition B

Let $V$ be a vector space over $\mathbb{R}$. A nonempty indexed set $S = \{v_1, v_2, ..., v_n\}$ of vectors in $V$ is *linearly dependent* if there are $c_1, c_2, ..., c_n \in \mathbb{R}$, not all zeros, such that

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0.$$

Otherwise, $S$ is linearly independent.

Theorem (A $\iff$ B)

Let $S = \{v_1, v_2, ..., v_n\}$ be an indexed set of vectors in $V$. Then $\text{Span}(S) = \text{Span}(S \setminus \{v_i\})$ for some $v_i \in S$ if and only if there are $c_1, c_2, ..., c_n \in \mathbb{R}$, not all zeros, such that

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Definition B

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Proof.

“⇒” Span(S) = Span(S\{v_i\}) ⇒ v_i ∈ Span(S\{v_i\}). That is, v_i is a linear combination of v_j for j ≠ i:

\[ v_i = \sum_{j \neq i} c_j v_j \]

for some \( c_j \in \mathbb{R} \). Therefore,

\[ \sum_{j \neq i} c_j v_j - v_i = 0 \Rightarrow c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0 \]

by letting \( c_i = -1 \). So \( v_1, v_2, \ldots, v_n \) are linearly dependent. \( \square \)
Proof of A $\iff$ B

“$\Leftarrow$”.

$v_1, v_2, \ldots, v_n$ are linearly dependent $\Rightarrow$ there exist $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0.$$

Suppose that $c_i \neq 0$ for some $i$. Then

$$c_i v_i = -\sum_{j \neq i} c_j v_j \Rightarrow v_i = -\sum_{j \neq i} \frac{c_j}{c_i} v_j$$

and $v_i \in \text{Span}(S \setminus \{v_i\})$. So

$$S \subset \text{Span}(S \setminus \{v_i\}) \Rightarrow \text{Span}(S) \subset \text{Span}(S \setminus \{v_i\})$$

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$\Box$
Proof of $A \iff B$

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Proof of \( A \Leftrightarrow B \)

“\( \Leftarrow \).”

\( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) are linearly dependent \( \Rightarrow \) there exist \( c_1, c_2, \ldots, c_n \), not all zero, such that

\[ c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n = 0. \]

Suppose that \( c_i \neq 0 \) for some \( i \). Then

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and \( \text{Span}(S) = \text{Span}(S \setminus \{\mathbf{v}_i\}) \).
The above theorem is equivalent to saying that \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) are linearly dependent if and only if one of \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) is a linear combination of the rest.

- \( \mathbf{0} \in S \Rightarrow S \) is linearly dependent; e.g., \( \{\mathbf{0}, \mathbf{v}\} \) is always linearly dependent since \( 1 \cdot \mathbf{0} + 0 \cdot \mathbf{v} = \mathbf{0} \).
- \( S' \subset S \) and \( S \) is linearly independent \( \Rightarrow \) \( S' \) is linearly independent; e.g., if \( \{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \) is linearly independent, then \( \{\mathbf{u}, \mathbf{v}\} \) is linearly independent.
- \( S \subset S' \) and \( S \) is linearly dependent \( \Rightarrow \) \( S' \) is linearly dependent; e.g., if \( \{\mathbf{u}, \mathbf{v}\} \) is linearly dependent, then \( \{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \) is linearly dependent for all \( \mathbf{w} \).
The above theorem is equivalent to saying that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent if and only if one of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a linear combination of the rest.

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**Theorem**

\( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n \) are linearly independent if and only if one of the following holds:

- The \( m \times n \) matrix
  \[
  A = \begin{bmatrix}
  \mathbf{v}_1 \\
  \mathbf{v}_2 \\
  \vdots \\
  \mathbf{v}_m
  \end{bmatrix}
  \]
  has rank \( m \).

- \( A^T \mathbf{x} = 0 \) has a unique solution.

- There exists an \( m \times m \) matrix \( B \) such that \( BA \) is a row echelon matrix without zero rows.

- There exists an \( n \times n \) matrix \( B \) such that \( AB = [I \ 0] \).
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Linear Dependence in $\mathbb{R}^n$

Theorem

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Xi Chen  
Linear Algebra II Lecture 4
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Examples of Linear Dependence in $\mathbb{R}^n$

- $(1, 1, 1, 1), (1, 2, 3, 4), (2, 3, 4, 5)$ are linearly dependent in $\mathbb{R}^4$ since

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} = 2 < 3.$$
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\end{bmatrix}
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$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ are linearly dependent if $m > n$.

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \neq 0$ are orthogonal to each other, i.e., $\langle \mathbf{v}_i, \mathbf{v}_j \rangle \neq 0$ if $i = j$ and $0$ if $i \neq j$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent since

$$AA^T = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T & \mathbf{v}_2^T & \ldots & \mathbf{v}_m^T \end{bmatrix} = [\langle \mathbf{v}_i, \mathbf{v}_j \rangle]$$

$\Rightarrow \text{rank}(AA^T) = m \Rightarrow \text{rank}(A) \geq \text{rank}(AA^T) = m$ and hence $\text{rank}(A) = m$. In particular, the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is linearly independent.
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Let $a_1, a_2, \ldots, a_n$ be $n$ distinct numbers. Then

$(1, a_1, a_1^2, \ldots, a_1^{n-1}), (1, a_2, a_2^2, \ldots, a_2^{n-1}), \ldots,$

$(1, a_n, a_n^2, \ldots, a_n^{n-1})$ are linearly independent in $\mathbb{R}^n$.

Otherwise, the $n \times n$ matrix

$$A = \begin{bmatrix}
1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\
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is singular so $Ax = 0$ for some $x \neq 0$:
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Let $a_1, a_2, \ldots, a_n$ be $n$ distinct numbers. Then $(1, a_1, a_1^2, \ldots, a_1^{n-1})$, $(1, a_2, a_2^2, \ldots, a_2^{n-1})$, ..., $(1, a_n, a_n^2, \ldots, a_n^{n-1})$ are linearly independent in $\mathbb{R}^n$. Otherwise, the $n \times n$ matrix

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\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix} = 0 \Rightarrow c_1 + c_2 a_i + \ldots + c_n a_i^{n-1} = 0
\]

for $i = 1, 2, \ldots, n$. Let

\[f(x) = c_1 + c_2 x + \ldots + c_n x^{n-1}\]

Then $f(a_1) = f(a_2) = \ldots = f(a_n) = 0$. So $f(x)$ is divisible by $(x - a_1)(x - a_2)\ldots(x - a_n)$, which is impossible since $\deg f \leq n - 1$. 
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Examples of Linear Dependence in $\mathbb{R}^n$

Let $A$ be an $n \times n$ matrix, $a_1, a_2, \ldots, a_m$ be $m$ distinct eigenvalues of $A$ with corresponding eigenvectors $v_1, v_2, \ldots, v_m$. Then $v_1, v_2, \ldots, v_m$ are linearly independent in $\mathbb{R}^n$. Otherwise, there exist $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m = 0.$$ 

Since $A v_1 = a_1 v_1$, $A v_2 = a_2 v_2$, ..., $A v_m = a_m v_m$,

$$A (c_1 v_1 + c_2 v_2 + \ldots + c_m v_m) = 0$$

$$\Rightarrow a_1 c_1 v_1 + a_2 c_2 v_2 + \ldots + a_m c_m v_m = 0$$

Multiplying by $A^k$, we obtain

$$a_1^k c_1 v_1 + a_2^k c_2 v_2 + \ldots + a_m^k c_m v_m = 0.$$
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$$\Rightarrow a_1 c_1 v_1 + a_2 c_2 v_2 + \ldots + a_m c_m v_m = 0$$

Multiplying by $A^k$, we obtain

$$a_1^k c_1 v_1 + a_2^k c_2 v_2 + \ldots + a_m^k c_m v_m = 0$$
Let $A$ be an $n \times n$ matrix, $a_1, a_2, \ldots, a_m$ be $m$ distinct eigenvalues of $A$ with corresponding eigenvectors $v_1, v_2, \ldots, v_m$. Then $v_1, v_2, \ldots, v_m$ are linearly independent in $\mathbb{R}^n$. Otherwise, there exist $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m = 0.$$ 

Since $A v_1 = a_1 v_1, A v_2 = a_2 v_2, \ldots, A v_m = a_m v_m,$

$$A(c_1 v_1 + c_2 v_2 + \ldots + c_m v_m) = 0$$

$$\Rightarrow a_1 c_1 v_1 + a_2 c_2 v_2 + \ldots + a_m c_m v_m = 0$$

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\[
c_1 v_1 + c_2 v_2 + \ldots + c_m v_m = 0.
\]
Since $Av_1 = a_1 v_1$, $Av_2 = a_2 v_2$, ..., $Av_m = a_m v_m$,
\[
A(c_1 v_1 + c_2 v_2 + \ldots + c_m v_m) = 0
\]
$\Rightarrow a_1 c_1 v_1 + a_2 c_2 v_2 + \ldots + a_m c_m v_m = 0$

Multiplying by $A^k$, we obtain
\[
a_1^k c_1 v_1 + a_2^k c_2 v_2 + \ldots + a_m^k c_m v_m = 0
\]
Examples of Linear Dependence in $\mathbb{R}^n$

\[
\begin{align*}
&c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_m \mathbf{v}_m = 0 \\
&a_1 c_1 \mathbf{v}_1 + a_2 c_2 \mathbf{v}_2 + \ldots + a_m c_m \mathbf{v}_m = 0 \\
&\vdots \\
&a_1^{m-1} c_1 \mathbf{v}_1 + a_2^{m-1} c_2 \mathbf{v}_2 + \ldots + a_m^{m-1} c_m \mathbf{v}_m = 0 \\
\end{align*}
\]

\[
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
a_1 & a_2 & \ldots & a_m \\
\vdots & \vdots & \ddots & \vdots \\
a_1^{m-1} & a_2^{m-1} & \ldots & a_m^{m-1}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_m
\end{pmatrix}
= 0
\]

Since $B$ is nonsingular, $c_1 \mathbf{v}_1 = c_2 \mathbf{v}_2 = \ldots = c_m \mathbf{v}_m = 0$. This is impossible since $\mathbf{v}_i \neq 0$ and $c_1, c_2, \ldots, c_m$ are not all zero.
Let $A$ be an $n \times n$ matrix. Then $1, A, A^2, \ldots, A^n$ are linearly dependent in $M_{n\times n}(\mathbb{R})$ since

\[ f(x) = \det(xI - A) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + x^n \]

be its characteristic polynomial. Then

\[ f(A) = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1} + A^n = 0 \]

e.g., for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^2 - 5A - 2I = 0 \Rightarrow A^2, A, I$ are linearly dependent.
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**Caley-Hamilton Theorem**

Let $A$ be an $n \times n$ matrix and

$$f(x) = \det(xI - A) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + x^n$$

be its characteristic polynomial. Then

$$f(A) = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1} + A^n = 0$$

e.g., for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^2 - 5A - 2I = 0 \Rightarrow A^2, A, I$ are linearly dependent.
Let $f_1(x), f_2(x), \ldots, f_m(x)$ be $m$ polynomials of degree $< n$:

$$f_i(x) = a_{i1} + a_{i2}x + \ldots + a_{in}x^{n-1}$$

for $i = 1, 2, \ldots, m$. Then $c_1 f_1(x) + c_2 f_2(x) + \ldots + c_m f_m(x) = 0$ if and only if

$$\begin{align*}
    a_{11}c_1 + a_{21}c_2 + \ldots + a_{m1}c_m &= 0 \\
    a_{12}c_1 + a_{22}c_2 + \ldots + a_{m2}c_m &= 0 \\
    \vdots + \vdots + \ddots + \vdots &= \vdots \\
    a_{1n}c_1 + a_{2n}c_2 + \ldots + a_{mn}c_m &= 0
\end{align*}$$

$$\leftrightarrow$$

$$\begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}^T
\begin{bmatrix}
    c_1 \\
    c_2 \\
    \vdots \\
    c_m
\end{bmatrix} = 0$$
Therefore, \( f_1(x), f_2(x), \ldots, f_m(x) \) are linearly independent if and only if

\[
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}
\]

is a square matrix of order \( m \times m \) and its rank is equal to \( m \).

Choose \( m \) distinct numbers \( x_1, x_2, \ldots, x_m \) and let

\[
B = \begin{bmatrix}
  f_1(x_1) & f_1(x_2) & \ldots & f_1(x_m) \\
  f_2(x_1) & f_2(x_2) & \ldots & f_2(x_m) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_m(x_1) & f_m(x_2) & \ldots & f_m(x_m)
\end{bmatrix}
\]
Therefore, \( f_1(x), f_2(x), \ldots, f_m(x) \) are linearly independent if and only if

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

has rank \( m \).

Choose \( m \) distinct numbers \( x_1, x_2, \ldots, x_m \) and let

\[
B = \begin{bmatrix}
  f_1(x_1) & f_1(x_2) & \cdots & f_1(x_m) \\
  f_2(x_1) & f_2(x_2) & \cdots & f_2(x_m) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_m(x_1) & f_m(x_2) & \cdots & f_m(x_m)
\end{bmatrix}
\]
Linear Dependence in $\mathbb{R}[x]$

Therefore, $\text{rank}(A) \geq \text{rank}(B)$. If $\text{rank}(B) = m$, then $f_1, f_2, \ldots, f_m$ are linearly independent. Caution: The converse fails.
Linear Dependence in $\mathbb{R}[x]$

Therefore, $\text{rank}(A) \geq \text{rank}(B)$. If $\text{rank}(B) = m$, then $f_1, f_2, \ldots, f_m$ are linearly independent. Caution: The converse fails.