Solutions for Math 225 Assignment #7

(1) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x, y) = (3x + 4y, 4x - 3y).$$

(a) Find the characteristic polynomial, eigenvalues and eigenvectors of $T$.

(b) Find a basis $B$ of $\mathbb{R}^2$ such that $[T]_{B,B}$ is a diagonal matrix.

Proof. Let $C$ be the standard basis of $\mathbb{R}^2$. Then

$$[T]_{C,C} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

and the characteristic polynomial of $T$ is

$$\det(xI - [T]_{C,C}) = x^2 - 25.$$ 

Therefore, $T$ has eigenvalues $\pm 5$ with eigenvectors

$$\{v : (5I - T)v = 0\} = \{(x, y) : x - 2y = 0\} = \text{Span}\{(2, 1)\}$$

and

$$\{v : (-5I - T)v = 0\} = \{(x, y) : 2x + y = 0\} = \text{Span}\{(1, -2)\}.$$ 

Let $B = \{(2, 1), (1, -2)\}$. Then

$$[T]_{B,B} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.$$ 

\(\Box\)

(2) Let $M_{m \times n}(\mathbb{R})$ be the vector space of $m \times n$ real matrices and let $T : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})$ be the linear transformation given by

$$T(A) = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} A$$

(a) Find the characteristic polynomial, eigenvalues and eigenvectors of $T$.

(b) Find a basis $B$ of $M_{2 \times 2}(\mathbb{R})$ such that $[T]_{B,B}$ is a diagonal matrix.

1http://www.math.ualberta.ca/~xichen/math22514w/hw7sol.pdf
Solution. We let $C$ be the basis
\[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0 
\end{bmatrix}
\]
of $M_{2 \times 2}(\mathbb{R})$. By
\[
\begin{bmatrix}
3 & 0 \\
4 & 0 
\end{bmatrix}
= 3
\begin{bmatrix}
1 & 0 \\
0 & 0 
\end{bmatrix}
+ 4
\begin{bmatrix}
0 & 0 \\
1 & 0 
\end{bmatrix}
+ 0
\begin{bmatrix}
0 & 1 \\
0 & 0 
\end{bmatrix}
+ 0
\begin{bmatrix}
0 & 0 \\
0 & 1 
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & 0 \\
1 & 0 
\end{bmatrix}
= 4
\begin{bmatrix}
1 & 0 \\
0 & 0 
\end{bmatrix}
- 3
\begin{bmatrix}
0 & 0 \\
1 & 0 
\end{bmatrix}
+ 0
\begin{bmatrix}
0 & 1 \\
0 & 0 
\end{bmatrix}
+ 0
\begin{bmatrix}
0 & 0 \\
0 & 1 
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & 1 \\
0 & 0 
\end{bmatrix}
= 0
\begin{bmatrix}
1 & 0 \\
0 & 0 
\end{bmatrix}
+ 0
\begin{bmatrix}
0 & 0 \\
1 & 0 
\end{bmatrix}
+ 3
\begin{bmatrix}
0 & 1 \\
0 & 0 
\end{bmatrix}
+ 4
\begin{bmatrix}
0 & 0 \\
0 & 1 
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & 0 \\
0 & 1 
\end{bmatrix}
= 0
\begin{bmatrix}
1 & 0 \\
0 & 0 
\end{bmatrix}
+ 0
\begin{bmatrix}
0 & 0 \\
1 & 0 
\end{bmatrix}
+ 4
\begin{bmatrix}
0 & 1 \\
0 & 0 
\end{bmatrix}
- 3
\begin{bmatrix}
0 & 0 \\
0 & 1 
\end{bmatrix}
\]
we obtain
\[
[T]_{C,C} = \begin{bmatrix}
3 & 4 \\
4 & -3 \\
3 & 4 \\
4 & -3 
\end{bmatrix}
\]
So the characteristic polynomial of $T$ is
\[
\det(xI - [T]_{C,C}) = \left( \det \left( xI - \begin{bmatrix}
3 & 4 \\
4 & -3 
\end{bmatrix} \right) \right)^2 = (x^2 - 25)^2
\]
and the eigenvalues of $T$ are $\pm 5$ with eigenvectors
\[
\{ A : (5I - T)A = 0 \} = \left\{ A : \begin{bmatrix}
2 & -4 \\
-4 & 8 
\end{bmatrix} A = 0 \right\}
= \text{Span} \left\{ \begin{bmatrix}
2 & 0 \\
1 & 0 
\end{bmatrix}, \begin{bmatrix}
0 & 2 \\
0 & 1 
\end{bmatrix} \right\}
\]
and
\[
\{ A : (-5I - T)A = 0 \} = \left\{ A : \begin{bmatrix}
-8 & -4 \\
-4 & -2 
\end{bmatrix} A = 0 \right\}
= \text{Span} \left\{ \begin{bmatrix}
1 & 0 \\
-2 & 0 
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
0 & -2 
\end{bmatrix} \right\}
\]
Let
\[
B = \left\{ \begin{bmatrix}
2 & 0 \\
1 & 0 
\end{bmatrix}, \begin{bmatrix}
0 & 2 \\
0 & 1 
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
-2 & 0 
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
0 & -2 
\end{bmatrix} \right\}.
Then
\[ [T]_{B,B} = \begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix}. \]

(3) Let \( V \) be a vector space of dimension \( n \) and \( T : V \to V \) be a linear transformation of rank 1. Show that the characteristic polynomial of \( T \) must be in the form of \( x^n - ax^{n-1} \) for some constant \( a \in \mathbb{R} \).

Proof. By Rank Theorem,
\[ \dim K(T) = \dim V - \text{rank}(T) = n - 1. \]

We choose a basis \( B = \{ v_1, v_2, ..., v_n \} \) of \( V \) with \( \{ v_2, v_3, ..., v_n \} \) a basis of \( K(T) \). Then
\[
\begin{align*}
T(v_1) &= a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \\
T(v_2) &= 0 \\
T(v_3) &= 0 \\
&\vdots \\
T(v_n) &= 0
\end{align*}
\]

and hence
\[
[T]_{B,B} = \begin{bmatrix}
a_1 & 0 & \ldots & 0 \\
a_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_n & 0 & \ldots & 0
\end{bmatrix}
\]

Therefore, the characteristic polynomial of \( T \) is
\[
\det(xI - [T]_{B,B}) = \det \begin{bmatrix}
x - a_1 & 0 & \ldots & 0 \\
-a_2 & x & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_n & 0 & \ldots & x
\end{bmatrix} = (x - a_1)x^{n-1} = x^n - a_1x^{n-1}.
\]

(4) Which of the following statements are true and which are false? Justify your answer.
(a) Let $T_1 : V \to V$ and $T_2 : V \to V$ be two linear transformations. If $v_1$ is an eigenvector of $T_1$ and $v_2$ is an eigenvector of $T_2$, then $v_1 + v_2$ is an eigenvector of $T_1 + T_2$.

Proof. False. For example, we let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations given by $T_1(x, y) = (x, 0)$ and $T_2(x, y) = (0, 2y)$, respectively, and let $v_1 = (1, 0)$ and $v_2 = (0, 1)$. Clearly, $v_1$ is an eigenvector of $T_1$ and $v_2$ is an eigenvector of $T_2$. But $(T_1 + T_2)(x, y) = (x, 2y)$ and $v_1 + v_2 = (1, 1)$ is not an eigenvector of $T_1 + T_2$. □

(b) Let $A$ and $B$ be two $n \times n$ invertible matrices. Then $AB$ and $BA$ have the same characteristic polynomial.

Proof. True since $AB$ and $BA$ are similar: $BA = A^{-1}(AB)A$. □

(c) Let $T : V \to V$ be a linear transformation. If $v$ is an eigenvector of $T$, it is also an eigenvector of $T^2$.

Proof. True since

$$T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v.$$ □

(d) Let $T : V \to V$ be a linear transformation. If $v$ is an eigenvector of $T^2$, it is also an eigenvector of $T^3$.

Proof. False. For example, we let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (x, -y)$ and let $v = (1, 1)$. Clearly, $v$ is an eigenvector of $T^2(x, y) = T(x, -y) = (x, -(x)) = (x, y)$ and it is not an eigenvector of $T^3(x, y) = T(T^2(x, y)) = T(x, y) = (x, -(y))$. □

(5) Let $V$ be the vector space of real polynomials of degree $\leq 3$ and $T : V \to V$ be the linear transformation given by

$$T(f(x)) = (x + 1)f'(x).$$

(a) Find the characteristic polynomial, eigenvalues and eigenvectors of $T$. 
(b) Find a basis $B$ of $V$ such that $[T]_{B,B}$ is a diagonal matrix.

Solution. Let $C = \{1, x, x^2, x^3\}$ be the standard basis of $V$. By

\[
T(1) = (x + 1)(1)' = 0 \\
T(x) = (x + 1)(x)' = 1 + x \\
T(x^2) = (x + 1)(x^2)' = 0 + 2x + 2x^2 \\
T(x^3) = (x + 1)(x^3)' = 0 + 0 + 3x^2 + 3x^3
\]

we obtain

\[
[T]_{C,C} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 2 & 3 \\
0 & 0 & 0 & 3
\end{bmatrix}.
\]

Therefore, the characteristic polynomial of $T$ is

\[
\det(xI - [T]_{C,C}) = x(x - 1)(x - 2)(x - 3)
\]

and $T$ has eigenvalues 0, 1, 2, 3.

Since $T((x + 1)^n) = (x + 1)((x + 1)^n)' = n(x + 1)^n$, $(x + 1)^n$ is an eigenvector of $T$ corresponding to eigenvalue $n$. Therefore, the eigenvectors of $T$ are 1, $x + 1$, $(x + 1)^2$, $(x + 1)^3$.

Let $B = \{1, x + 1, (x + 1)^2, (x + 1)^3\}$. Then

\[
[T]_{B,B} = \begin{bmatrix}
0 & 1 & 2 & 3
\end{bmatrix}.
\]

\[\square\]

(6) Let $V$ be a real vector space of dimension 2014 and $T : V \to V$ be the linear transformation defined by

\[
T(v_1) = v_2, T(v_2) = v_3, \ldots, T(v_{2013}) = v_{2014}, T(v_{2014}) = v_1
\]

for a basis $\{v_1, v_2, \ldots, v_{2014}\}$ of $V$. Find all the real eigenvalues and eigenvectors of $T$.

Solution. Let

\[
v = a_1v_1 + a_2v_2 + \ldots + a_{2014}v_{2014}
\]
be an eigenvector of $T$ corresponding to $\lambda$. Then

\[
T(v) = \lambda v \iff T(a_1 v_1 + a_2 v_2 + \ldots + a_{2014} v_{2014}) = \lambda (a_1 v_1 + a_2 v_2 + \ldots + a_{2014} v_{2014})
\]

\[
\iff a_1 T(v_1) + a_2 T(v_2) + \ldots + a_{2013} T(v_{2013}) + a_{2014} T(v_{2014}) = \lambda a_1 v_1 + \lambda a_2 v_2 + \ldots + \lambda a_{2013} v_{2013} + \lambda a_{2014} v_{2014}
\]

\[
\iff a_1 v_2 + a_2 v_3 + \ldots + a_{2013} v_{2014} + a_{2014} v_1 = \lambda a_1 v_1 + \lambda a_2 v_2 + \ldots + \lambda a_{2013} v_{2013} + \lambda a_{2014} v_{2014}.
\]

Therefore,

\[
(a_{2014} - \lambda a_1) v_1 + (a_1 - \lambda a_2) v_2 + \ldots + (a_{2013} - \lambda a_{2014}) v_{2014} = 0.
\]

And since $v_1, v_2, \ldots, v_{2014}$ are linearly independent,

\[
a_{2014} - \lambda a_1 = a_1 - \lambda a_2 = \ldots = a_{2013} - \lambda a_{2014} = 0
\]

i.e.,

\[
a_{2014} = \lambda a_1 \text{ and } a_k = \lambda a_{k+1} \text{ for } k = 1, 2, \ldots, 2013.
\]

If $a_{2014} = 0$, then

\[
a_{2014} = 0 \implies a_{2013} = \lambda a_{2014} = 0 \implies \ldots \implies a_1 = 0
\]

and hence $v = 0$, which is impossible. So $a_{2014} \neq 0$.

By $a_k = \lambda a_{k+1}$, we obtain

\[
a_1 = \lambda a_2 = \lambda (\lambda a_3) = \lambda^2 a_3 = \ldots = \lambda^{2013} a_{2014}.
\]

On the other hand, $a_{2014} = \lambda a_1$ and hence

\[
a_{2014} = \lambda (\lambda^{2013} a_{2014}) = \lambda^{2014} a_{2014}.
\]

And since $a_{2014} \neq 0$, $\lambda^{2014} = 1$ and hence $\lambda = \pm 1$.

So $T$ has two eigenvalues 1 and $-1$ with eigenvectors

\[
\{ v : (I - T)v = 0 \} = \text{Span}\{ v_1 + v_2 + \ldots + v_{2014} \}
\]

\[
= \text{Span} \left\{ \sum_{k=1}^{2014} v_k \right\}
\]

and

\[
\{ v : (-I - T)v = 0 \} = \text{Span}\{ v_1 - v_2 + \ldots - v_{2014} \}
\]

\[
= \text{Span} \left\{ \sum_{k=1}^{2014} (-1)^{k+1} v_k \right\}.
\]

$\square$
(7) We call a linear transformation \( T : V \to V \) a projection if there are subspaces \( W_1 \) and \( W_2 \) of \( V \) such that \( V = W_1 + W_2 \), \( T(w_1) = 0 \) for all \( w_1 \in W_1 \) and \( T(w_2) = w_2 \) for all \( w_2 \in W \).

Show that \( T \) is a projection if and only if \( T^2 = T \).

Proof. Suppose that \( T \) is a projection. Then
\[
T(w_1 + w_2) = T(w_1) + T(w_2) = w_2
\]
and
\[
T^2(w_1 + w_2) = T(w_2) = w_2 = T(w_1 + w_2)
\]
for all \( w_1 \in W_1 \) and \( w_2 \in W_2 \). And since \( V = W_1 + W_2 \), \( T^2(v) = T(v) \) for all \( v \in V \), i.e., \( T^2 = T \).

Suppose that \( T^2 = T \). We let \( W_1 = K(T) \) and \( W_2 = R(T) \).

For every \( v \in V \), we have
\[
v = (v - T(v)) + T(v).
\]
Since \( T^2 = T \),
\[
T(v - T(v)) = T(v) - T^2(v) = 0
\]
and it follows that \( v - T(v) \in K(T) = W_1 \). Therefore, we have
\[
v = \underbrace{(v - T(v)) + T(v)}_{\in W_1} = \underbrace{v}_{\in W_2}
\]
for every \( v \in V \) and hence \( V = W_1 + W_2 \).

For each \( w_1 \in W_1 = K(T) \), \( T(w_1) = 0 \).

For each \( w_2 \in W_2 = R(T) \), \( w_2 = T(v) \) for some \( v \in V \).

Therefore,
\[
T(w_2) = T(T(v)) = T^2(v) = T(v) = w_2.
\]

\( \square \)

(8) Let \( T : V \to V \) be a projection, as defined in the previous problem. Show that \( T \) has no eigenvalues other than 0 and 1.

Proof. Let \( v \) be an eigenvector of \( T \) corresponding to an eigenvalue \( \lambda \). Since \( T^2 = T \), \( (T^2 - T)(v) = 0 \) and hence
\[
(T^2 - T)(v) = (\lambda^2 - \lambda)v = 0.
\]
And since \( v \neq 0 \), \( \lambda^2 - \lambda = 0 \) and hence \( \lambda = 0 \) or 1. Consequently, \( T \) has no eigenvalues other than 0 and 1. \( \square \)