Math 225 Assignment #7
Due Mar. 14, 2014

(1) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x, y) = (3x + 4y, 4x - 3y).$$

(a) Find the characteristic polynomial, eigenvalues and eigenvectors of $T$.
(b) Find a basis $B$ of $\mathbb{R}^2$ such that $[T]_{B,B}$ is a diagonal matrix.

(2) Let $M_{m \times n}(\mathbb{R})$ be the vector space of $m \times n$ real matrices and let $T : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})$ be the linear transformation given by

$$T(A) = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} A$$

(a) Find the characteristic polynomial, eigenvalues and eigenvectors of $T$.
(b) Find a basis $B$ of $M_{2 \times 2}(\mathbb{R})$ such that $[T]_{B,B}$ is a diagonal matrix.

(3) Let $V$ be a vector space of dimension $n$ and $T : V \to V$ be a linear transformation of rank 1. Show that the characteristic polynomial of $T$ must be in the form of $x^n - ax^{n-1}$ for some constant $a \in \mathbb{R}$.

(4) Which of the following statements are true and which are false? Justify your answer.

(a) Let $T_1 : V \to V$ and $T_2 : V \to V$ be two linear transformations. If $v_1$ is an eigenvector of $T_1$ and $v_2$ is an eigenvector of $T_2$, then $v_1 + v_2$ is an eigenvector of $T_1 + T_2$.
(b) Let $A$ and $B$ be two $n \times n$ invertible matrices. Then $AB$ and $BA$ have the same characteristic polynomial.
(c) Let $T : V \to V$ be a linear transformation. If $v$ is an eigenvector of $T$, it is also an eigenvector of $T^2$.
(d) Let $T : V \to V$ be a linear transformation. If $v$ is an eigenvector of $T^2$, it is also an eigenvector of $T^3$.

(5) Let $V$ be the vector space of real polynomials of degree $\leq 3$ and $T : V \to V$ be the linear transformation given by

$$T(f(x)) = (x + 1)f'(x).$$

(a) Find the characteristic polynomial, eigenvalues and eigenvectors of $T$.
(b) Find a basis $B$ of $V$ such that $[T]_{B,B}$ is a diagonal matrix.
(6) Let $V$ be a real vector space of dimension 2014 and $T : V \to V$ be the linear transformation defined by

$T(v_1) = v_2, T(v_2) = v_3, \ldots, T(v_{2013}) = v_{2014}, T(v_{2014}) = v_1$

for a basis $\{v_1, v_2, \ldots, v_{2014}\}$ of $V$. Find all the real eigenvalues and eigenvectors of $T$.

(7) We call a linear transformation $T : V \to V$ a projection if there are subspaces $W_1$ and $W_2$ of $V$ such that $V = W_1 + W_2$, $T(w_1) = 0$ for all $w_1 \in W_1$ and $T(w_2) = w_2$ for all $w_2 \in W$. Show that $T$ is a projection if and only if $T^2 = T$.

(8) Let $T : V \to V$ be a projection, as defined in the previous problem. Show that $T$ has no eigenvalues other than 0 and 1.