(1) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by

$$T(x, y, z) = (x + y + z, x - z).$$

Find the matrix $[T]_{B_1, B_2}$ representing $T$ for

(a) $B_1$ and $B_2$ the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$

(b) $B_1 = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ and $B_2 = \{(3, 4), (4, 5)\}$.

(2) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation satisfying

$$T(e_1) = e_2, T(e_2) = e_3$$

and $K(T) = \text{Span}\{ (1, 1, 1) \}$, where $K(T)$ is the kernel of $T$.

(a) Find the matrix $[T]_{B,B}$ representing $T$ under the standard basis $B$ of $\mathbb{R}^3$.

(b) Find the range $R(T)$ of $T$.

(3) Let $V = \{ f(x) \in \mathbb{R}[x] : \deg f < 3 \}$ be the vector space of real polynomials in $x$ of degree $< 3$ and let $T_1 : V \rightarrow V$ and $T_2 : V \rightarrow V$ be two linear transformations given by

$$T_1(f(x)) = xf'(x) \text{ and } T_2(f(x)) = f(1 - x).$$

(a) Find $T_1 + T_2$ and its matrix representation $[T_1 + T_2]_{B,B}$.

(b) Find $T_1 \circ T_2$ and its matrix representation $[T_1 \circ T_2]_{B,B}$.

(c) Find $T_2 \circ T_1$ and its matrix representation $[T_2 \circ T_1]_{B,B}$.

(d) Verify that

$$[T_1 + T_2]_{B,B} = [T_1]_{B,B} + [T_2]_{B,B}$$

$$[T_1 \circ T_2]_{B,B} = [T_1]_{B,B}[T_2]_{B,B}$$

$$[T_2 \circ T_1]_{B,B} = [T_2]_{B,B}[T_1]_{B,B}$$

Here $B$ is the basis $\{1, x, x^2\}$.

(4) Which of the following statements are true and which are false? Justify your answer.

(a) Let $T : V \rightarrow W$ and $S : U \rightarrow V$ be two linear transformations. Then the kernel $K(S)$ of $S$ is contained in the kernel $K(T \circ S)$ of $T \circ S$.

(b) Let $T : V \rightarrow W$ and $S : U \rightarrow V$ be two linear transformations. Then the range $R(T)$ of $T$ is contained in the range $R(T \circ S)$ of $T \circ S$.

(c) Let $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ be two linear transformations. Then $R(T_1 + T_2) = R(T_1) + R(T_2)$. \hfill 1
(d) If \( T : V \to V \) is a linear transformation satisfying \( R(T) = R(T^2) \), then \( R(T^{2013}) = R(T^{2014}) \), where
\[
T^n = T \circ T \circ \ldots \circ T.
\]

(5) Find a linear transformation \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) such that
\[
K(T) = \{0\} \text{ and } R(T) \neq \mathbb{R}[x].
\]

(6) Find the kernels and ranges of the following linear transformations:
(a) \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) given by \( T(x, y) = (x, y, x + y) \);
(b) \( T : M_{n \times n}(\mathbb{R}) \to M_{n \times n}(\mathbb{R}) \) given by \( T(A) = A + A^T \);
(c) \( T : \mathbb{R}[x] \to \mathbb{R}^2 \) given by \( T(f(x)) = (f(1), f'(2)) \);
(d) \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) given by \( T(f(x)) = xf'(x) + f(x + 1) \).

(7) Let \( M_{m \times n}(\mathbb{R}) \) be the vector space of \( m \times n \) real matrices and \( T : M_{m \times n}(\mathbb{R}) \to M_{m \times n}(\mathbb{R}) \) be the linear transformation given by
\[
T(A) = PAQ
\]
for some \( m \times m \) matrix \( P \) and \( n \times n \) matrix \( Q \). Show that
\[
K(T) = \{0\}
\]
if and only if both \( P \) and \( Q \) are nonsingular.

(8) Let \( T : V \to W \) and \( S : U \to V \) be two linear transformations. Show that \( T \circ S = 0 \) (i.e. \( T(S(u)) = 0 \) for all \( u \in U \)) if and only if \( R(S) \subset K(T) \).