Math 225 Assignment #3
Due Jan 31, 2014

(1) Let
\[ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & 3 & 1 & 5 \end{bmatrix}. \]
Find a basis and the dimension of a) \text{Nul}(A) b) \text{Row}(A) c) \text{Col}(A)

(2) Let \( M_{3 \times 3}(\mathbb{R}) \) be the vector space of \( 3 \times 3 \) real matrices. Find a basis and the dimension of
(a) the subspace of \( M_{3 \times 3}(\mathbb{R}) \) consisting of \( A \) satisfying
\[ A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]
(b) the subspace of \( M_{3 \times 3}(\mathbb{R}) \) consisting of \( A \) satisfying
\[ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \]

(3) Let \( \mathbb{R}[x] \) be the vector space of all real polynomials in \( x \). Find a basis and the dimension of
(a) the subspace of \( \mathbb{R}[x] \) consisting of \( f(x) \) satisfying \( \deg f \leq 3 \) and \( f(0) + f(1) = 0 \);
(b) the subspace of \( \mathbb{R}[x] \) consisting of \( f(x) \) satisfying \( \deg f \leq n \) and \( f'''(2) = 0 \) for some \( n \geq 3 \);
(c) the subspace of \( \mathbb{R}[x] \) consisting of \( f(x) \) satisfying \( \deg f \leq n \) and \( f(0) = f(1) = f(2) \) for some \( n \geq 3 \).

(4) Which of the following statements are true and which are false? Justify your answer.
(a) If \( \{ \mathbf{u}, \mathbf{v}, \mathbf{w} \} \) is a basis of \( V \), so is \( \{ \mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{w} + \mathbf{u} \} \).
(b) If \( \{ \mathbf{u}, \mathbf{v}, \mathbf{w} \} \) is a basis of \( V \), so is \( \{ \mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u} \} \).
(c) If \( A \) is a nonsingular \( 2 \times 2 \) real matrix, then \( I, A, A^{-1} \) are linearly dependent in \( M_{2 \times 2}(\mathbb{R}) \).
(d) If \( A \) is an \( n \times n \) matrix such that \( \text{Row}(A) = \text{Row}(A^2) \), then \( \text{Row}(A^{2013}) = \text{Row}(A^{2014}) \).

(5) Let \( A \) be an \( n \times n \) matrix. Show that if \( \text{Nul}(A) = \text{Nul}(A^2) \), then \( \text{Nul}(A^i) = \text{Nul}(A^j) \) for all \( i, j \geq 1 \).

(6) Which of the following maps are linear transformations and which are not? Justify your answer.
(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x, y) = (x, y, x + y)$;
(b) $T : M_{n \times n}(\mathbb{R}) \to M_{n \times n}(\mathbb{R})$ given by $T(A) = A A^T$;
(c) $T : \mathbb{R}[x] \to \mathbb{R}^2$ given by $T(f(x)) = (f(1), f'(2))$;
(d) $T : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $T(f(x)) = x f'(x) + f(x + 1)$.

(7) Let $\mathbb{R}^{m \times n}$ be the vector space of $m \times n$ real matrices. Let $T : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$ be the map given by
\[ T(A) = P A Q \]
for some $m \times m$ matrix $P$ and $n \times n$ matrix $Q$. Show that $T$ is a linear transformation.

(8) Let $\mathbb{R}^{2 \times 2}$ be the vector space of $2 \times 2$ real matrices and let $T : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2}$ be the map given by
\[ T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + d & b + c \\ b - c & a - d \end{bmatrix}. \]
(a) Show that $T$ is a linear transformation.
(b) Fixing a basis $B = \{ E_{11}, E_{12}, E_{21}, E_{22} \}$ of $\mathbb{R}^{2 \times 2}$ where
\[ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \]
find the matrix $[T]_{B,B}$ representing $T$ under $B$.
(c) Do there exist $2 \times 2$ matrices $P$ and $Q$ such that $T(A) = P A Q$ for all $A \in \mathbb{R}^{2 \times 2}$? Justify your answer.