Solutions for Math 225 Assignment #2

(1) Determine the values of $h$ and $k$ such that the solution set of the following system of linear equations (i) is empty (ii) contains a unique solution (iii) contains infinitely many solutions:

\[
\begin{align*}
    a) \quad \begin{cases}
                 x_1 - x_2 + x_3 &= k \\
                2x_1 - 3x_2 + x_3 &= 1 \\
                hx_1 + 2x_2 + x_3 &= 2
              \end{cases}
\end{align*}
\]

**Solution.** We apply row reduction to the augmented matrix

\[
\begin{bmatrix}
  1 & -1 & 1 & k \\
  2 & -3 & 1 & 1 \\
  h & 2 & 1 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  -1 & 2 & 0 & k - 1 \\
  2 & -3 & 1 & 1 \\
  h - 2 & 5 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  -\frac{1}{2} & 1 & 0 & \frac{k - 1}{2} \\
  2 & -3 & 1 & 1 \\
  h - 2 & 5 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  -\frac{1}{2} & 1 & 0 & \frac{k - 1}{2} \\
  2 & -3 & 1 & 1 \\
  h + \frac{1}{2} & 0 & 0 & \frac{7}{2} - \frac{5}{2}k
\end{bmatrix}
\]

Therefore, the system has a unique solution if

\[h + \frac{1}{2} \neq 0 \iff h \neq -\frac{1}{2}.\]

It has infinitely many solutions if

\[h + \frac{1}{2} = \frac{7}{2} - \frac{5}{2}k = 0 \iff h = -\frac{1}{2} \quad \text{and} \quad k = \frac{7}{5}.
\]

It has no solution if

\[h + \frac{1}{2} = 0 \quad \text{and} \quad \frac{7}{2} - \frac{5}{2}k \neq 0 \iff h = -\frac{1}{2} \quad \text{and} \quad k \neq \frac{7}{5}.
\]

\[\square\]

\[
\begin{align*}
    b) \quad \begin{cases}
                 x_1 + x_2 + hx_3 &= 1 \\
                x_1 + hx_2 + x_3 &= 1 \\
                hx_1 + x_2 + x_3 &= k
              \end{cases}
\end{align*}
\]

\[\text{http://www.math.ualberta.ca/~xichen/math22514w/hw2sol.pdf}\]
Solution. We apply row reduction to the augmented matrix

\[
\begin{bmatrix}
1 & 1 & h & 1 \\
1 & h & 1 & 1 \\
h & 1 & 1 & k
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & h & 1 \\
0 & h-1 & 1-h & 0 \\
0 & 1-h & 1-h^2 & k-h
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & h & 1 \\
0 & h-1 & 1-h & 0 \\
0 & 0 & 2-h-h^2 & k-h
\end{bmatrix}
\]

Therefore, the system has a unique solution if

\[h-1 \neq 0 \text{ and } 2-h-h^2 \neq 0 \iff h \neq -2,1\]

It has infinitely many solutions if

\[h = -2 \text{ or } 1 \text{ and } k-h = 0 \iff (h,k) = (-2,-2) \text{ or } (1,1)\]

It has no solution if

\[h = -2 \text{ or } 1 \text{ and } k-h \neq 0 \iff \begin{cases} h = -2 \\ k \neq -2 \end{cases} \text{ or } \begin{cases} h = 1 \\ k \neq 1 \end{cases}\]

\(□\)

(2) Find the characteristic polynomials, eigenvalues and eigenvectors of the following matrices:

a) \[
\begin{bmatrix}
1 & 2 \\
5 & 4
\end{bmatrix}
\]  

Answer. a) Characteristic polynomial: \(x^2-5x-6\). Eigenvalues: -1 and 6. Eigenvalues: (1, -1) and (2, 5).

b) \[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

Answer. b) Characteristic polynomial: \(x^3-3x-2\). Eigenvalues: -1 and 2. Eigenvalues: (1, -1, 0), (1, 0, -1) and (1, 1, 1). \(□\)

(3) Which of the following statements are true and which are false? Justify your answer.

(a) Let \(\mathbf{u}, \mathbf{v}, \mathbf{w}\) be three vectors in \(\mathbb{R}^n\). If \(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}\) is linearly independent, then each of \(\{\mathbf{u}, \mathbf{v}\}, \{\mathbf{v}, \mathbf{w}\}\) and \(\{\mathbf{u}, \mathbf{w}\}\) is linearly independent.

Proof. True. A subset of a linear independent set is linearly independent. \(□\)
(b) Let \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) be three vectors in \( \mathbb{R}^n \). If each of \( \{ \mathbf{u}, \mathbf{v} \} \), \( \{ \mathbf{v}, \mathbf{w} \} \) and \( \{ \mathbf{u}, \mathbf{w} \} \) is linearly independent, then \( \{ \mathbf{u}, \mathbf{v}, \mathbf{w} \} \) is linearly independent.

Proof. False. For example, let \( \mathbf{u} = (1, 0) \), \( \mathbf{v} = (0, 1) \) and \( \mathbf{w} = (1, 1) \in \mathbb{R}^2 \). Each of \( \{ \mathbf{u}, \mathbf{v} \} \), \( \{ \mathbf{v}, \mathbf{w} \} \) and \( \{ \mathbf{u}, \mathbf{w} \} \) is linearly independent but \( \{ \mathbf{u}, \mathbf{v}, \mathbf{w} \} \) is linearly dependent. \( \square \)

(c) \( \text{Nul}(A) \subset \text{Nul}(BA) \) for all \( m \times n \) matrices \( A \) and \( l \times m \) matrices \( B \).

Proof. True. For every \( \mathbf{v} \in \text{Nul}(A) \), \( A\mathbf{x} = 0 \). Therefore, \( BA\mathbf{x} = 0 \) and \( \mathbf{x} \in \text{Nul}(BA) \). \( \square \)

(d) \( \text{Row}(A) \subset \text{Row}(BA) \) for all \( m \times n \) matrices \( A \) and \( l \times m \) matrices \( B \).

Proof. False. For example, we let \( B = 0 \) and \( A \neq 0 \). Then \( \text{Row}(A) \not\subseteq \text{Row}(BA) = \text{Row}(0) = \{0\} \). \( \square \)

(4) Let \( A \) be a \( 3 \times 3 \) matrix with eigenvalues 1, 2, 3. Find the eigenvalues of (a) \( A^3 + A - I \) (b) \( A - A^{-1} \). Justify your answer.

Solution. Such \( A \) can be diagonalized: there exists \( 3 \times 3 \) invertible matrix \( P \) such that
\[
D = P^{-1}AP = \begin{bmatrix} 1 & 2 \\ & 3 \end{bmatrix}
\]
So
\[
P^{-1}(A^3 + A - I)P = D^3 + D - I = \begin{bmatrix} 1 & 9 \\ & 29 \end{bmatrix}
\]
and
\[
P^{-1}(A - A^{-1})P = D - D^{-1} = \begin{bmatrix} 0 & 3/2 \\ & 8/3 \end{bmatrix}
\]
Therefore, \( A^3 + A - I \) has eigenvalues 1, 9, 29 and \( A - A^{-1} \) has eigenvalues 0, 3/2, 8/3. \( \square \)
(5) Let \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) be three vectors in \( \mathbb{R}^3 \) and let \( A = [a_{ij}] \) be the \( 3 \times 3 \) matrix with entries \( a_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle \) for \( i, j = 1, 2, 3 \). Show that \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are linearly independent if and only if \( A \) is nonsingular.

**Proof.** Let

\[
B = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix}
\]

where \( \mathbf{v}_i \) are regarded as row vectors. Since \( \mathbf{v}_i \mathbf{v}_j^T = \langle \mathbf{v}_i, \mathbf{v}_j \rangle \),

\[
BB^T = [\langle \mathbf{v}_i, \mathbf{v}_j \rangle] = A.
\]

Therefore,

\[ A \text{ nonsingular} \iff B \text{ nonsingular} \iff \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \text{ are linearly independent}. \]

\[ \square \]

(6) Prove the following:

(a) Every square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

(b) Let \( W \) be the vector space of \( n \times n \) matrices and let \( U \) and \( V \) be the subspaces of \( W \) consisting of symmetric and skew-symmetric matrices, respectively. Then

\[ W = U + V. \]

**Proof.** Let \( A \) be a square matrix and let

\[
B = \frac{1}{2}(A + A^T) \quad \text{and} \quad C = \frac{1}{2}(A - A^T).
\]

Then \( A = B + C. \)

Since

\[
B^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A + A^T) = B
\]

\( B \) is symmetric. And since

\[
C^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T) = -\frac{1}{2}(A - A^T) = -C
\]

\( C \) is skew-symmetric. So every square matrix is the sum of a symmetric matrix and a skew-symmetric matrix.

For every \( A \in W \), \( A = B + C \) where \( B \) is symmetric and \( C \) is skew-symmetric, i.e., \( B \in U \) and \( C \in V \). Therefore, \( W = U + V. \)

\[ \square \]
(7) Let \( \mathbb{R}[x] \) be the vector space of all real polynomials in \( x \). Determine whether the following subsets of \( \mathbb{R}[x] \) are linearly independent. Justify your answer.

(a) A set of four quadratic polynomials.

Proof. This is linearly dependent. Let \( W \) be the subspace of \( \mathbb{R}[x] \) consisting of polynomials of degree \( \leq 2 \). Then \( W \) has a basis \( \{1, x, x^2\} \) and hence dimension \( \dim W = 3 \). Let \( f_1(x), f_2(x), f_3(x), f_4(x) \) be four polynomials of degree 2. Then \( f_1, f_2, f_3, f_4 \in W \). Since \( \dim W = 3 \), \( f_1, f_2, f_3, f_4 \) must be linearly dependent. \( \square \)

(b) \( \{1, x - 1, (x - 1)^2, (x - 1)^3\} \).

Proof. This is linearly independent. Since \( \deg(x - 1)^3 > \deg(x - 1)^2 > \deg(x - 1) > \deg 1 = 0 \), this is a special case of (d) (see the argument below). \( \square \)

(c) \( \{(x - 1)(x - 2)(x - 3), x(x - 1)(x - 3), x(x - 2)(x - 3), x(x - 1)(x - 2)\} \).

Proof. This is linearly independent. We evaluate the polynomials at 0, 1, 2, 3. Since

\[
\begin{bmatrix}
f_1(0) & f_1(1) & f_1(2) & f_1(3) \\
f_2(0) & f_2(1) & f_2(2) & f_2(3) \\
f_3(0) & f_3(1) & f_3(2) & f_3(3) \\
f_4(0) & f_4(1) & f_4(2) & f_4(3)
\end{bmatrix} = \begin{bmatrix}
-6 & 2 & -2 & 6
\end{bmatrix}
\]

is nonsingular,

\[f_1(x) = (x - 1)(x - 2)(x - 3)\]
\[f_2(x) = x(x - 2)(x - 3)\]
\[f_3(x) = x(x - 1)(x - 3)\]
\[f_4(x) = x(x - 1)(x - 2)\]

are linearly independent. \( \square \)

(d) \( \{f_1(x), f_2(x), \ldots, f_n(x)\} \), where

\[\deg f_1 > \deg f_2 > \ldots > \deg f_n \geq 0.\]
Proof. This is linearly independent. Let \( m = \deg f_1 \),
\[
f_i(x) = a_{i1}x^m + a_{i2}x^{m-1} + \ldots + a_{i,m+1}
\]
for \( i = 1, 2, \ldots, n \) and
\[
A = \begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1,m+1} \\
a_{21} & a_{22} & \ldots & a_{2,m+1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{n,m+1}
\end{bmatrix}.
\]
Since \( a_{ij} = 0 \) for \( j \leq m - \deg f_i \), \( a_{ij} \neq 0 \) for \( j = m + 1 - \deg f_i \) and
\[
m + 1 - \deg f_1 < m + 1 - \deg f_2 < \ldots < m + 1 - \deg f_n
\]
\( A \) is a row echelon matrix. And since \( \deg f_i \geq 0 \) for all \( i \), \( A \) has no zero rows. Therefore, \( \text{rank}(A) = n \) and \( f_1, f_2, \ldots, f_n \) are linearly independent. \( \square \)

(8) Let \( W_1 \) and \( W_2 \) be two subspaces of a vector space \( V \). Show that \( W_1 \cup W_2 \) is a subspace of \( V \) if and only if \( W_1 \subset W_2 \) or \( W_2 \subset W_1 \).

Proof. \( \Leftarrow \): If \( W_1 \subset W_2 \) or \( W_2 \subset W_1 \), \( W_1 \cup W_2 = W_2 \) or \( W_1 \) and hence \( W_1 \cup W_2 \) is a subspace of \( V \).

\( \Rightarrow \): Suppose that \( W_1 \cup W_2 \) is a subspace of \( V \), \( W_1 \not\subset W_2 \) and \( W_2 \not\subset W_1 \). Then there exist \( v_1 \in W_1 \) and \( v_2 \in W_2 \) such that \( v_1 \not\in W_2 \) and \( v_2 \not\in W_1 \).

Since \( v_1, v_2 \in W_1 \cup W_2 \) and \( W_1 \cup W_2 \) is a subspace,
\[
v_1 + v_2 \in W_1 \cup W_2.
\]
So either \( v_1 + v_2 \in W_1 \) or \( v_1 + v_2 \in W_2 \).

If \( v_1 + v_2 \in W_1 \),
\[
v_2 = (v_1 + v_2) - v_1 \in W_1
\]
since \( v_1 \in W_1 \) and \( W_1 \) is a subspace of \( V \). Contradiction.

If \( v_1 + v_2 \in W_2 \),
\[
v_1 = (v_1 + v_2) - v_2 \in W_2
\]
since \( v_2 \in W_2 \) and \( W_2 \) is a subspace of \( V \). Contradiction. \( \square \)