(1) Determine the values of $h$ and $k$ such that the solution set of the following system of linear equations (i) is empty (ii) contains a unique solution (iii) contains infinitely many solutions:

- **a)** \[
\begin{align*}
x_1 - x_2 + x_3 &= k \\
2x_1 - 3x_2 + x_3 &= 1 \\
hx_1 + 2x_2 + x_3 &= 2
\end{align*}
\]

- **b)** \[
\begin{align*}
x_1 + x_2 + hx_3 &= 1 \\
x_1 + hx_2 + x_3 &= 1 \\
hx_1 + x_2 + x_3 &= k
\end{align*}
\]

(2) Find the characteristic polynomials, eigenvalues and eigenvectors of the following matrices:

- **a)** \[
\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}
\]
- **b)** \[
\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}
\]

(3) Which of the following statements are true and which are false? Justify your answer.

- (a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors in $\mathbb{R}^n$. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then each of $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{v}, \mathbf{w}\}$ and $\{\mathbf{u}, \mathbf{w}\}$ is linearly independent.
- (b) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors in $\mathbb{R}^n$. If each of $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{v}, \mathbf{w}\}$ and $\{\mathbf{u}, \mathbf{w}\}$ is linearly independent, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.
- (c) $\text{Nul}(A) \subset \text{Nul}(BA)$ for all $m \times n$ matrices $A$ and $l \times m$ matrices $B$.
- (d) $\text{Row}(A) \subset \text{Row}(BA)$ for all $m \times n$ matrices $A$ and $l \times m$ matrices $B$.

(4) Let $A$ be a $3 \times 3$ matrix with eigenvalues 1, 2, 3. Find the eigenvalues of (a) $A^3 + A - I$ (b) $A - A^{-1}$. Justify your answer.

(5) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three vectors in $\mathbb{R}^3$ and let $A = [a_{ij}]$ be the $3 \times 3$ matrix with entries $a_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$ for $i, j = 1, 2, 3$. Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent if and only if $A$ is nonsingular.

(6) Prove the following:
(a) Every square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

(b) Let $W$ be the vector space of $n \times n$ matrices and let $U$ and $V$ be the subspaces of $W$ consisting of symmetric and skew-symmetric matrices, respectively. Then

$$W = U + V.$$ 

(7) Let $\mathbb{R}[x]$ be the vector space of all real polynomials in $x$. Determine whether the following subsets of $\mathbb{R}[x]$ are linearly independent. Justify your answer.

(a) A set of four quadratic polynomials.
(b) $\{1, x - 1, (x - 1)^2, (x - 1)^3\}$.
(c) $\{(x - 1)(x - 2)(x - 3), x(x - 1)(x - 3), x(x - 2)(x - 3), x(x - 1)(x - 2)\}$.
(d) $\{f_1(x), f_2(x), \ldots, f_n(x)\}$, where $\deg f_1 > \deg f_2 > \ldots > \deg f_n \geq 0$.

(8) Let $W_1$ and $W_2$ be two subspaces of a vector space $V$. Show that $W_1 \cup W_2$ is a subspace of $V$ if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. 