Solutions for Math 225 Assignment #1

(1) Solve the following linear equations using echelon form:

\begin{align*}
a) & \begin{cases} 
    x_1 - x_2 + x_3 + x_4 = 0 \\
    2x_1 - 3x_2 + x_3 = 0 \\
    2x_2 + x_3 + x_4 = 0 
\end{cases} \\
b) & \begin{cases} 
    x_1 - x_2 + x_3 = 1 \\
    2x_1 - 3x_2 + x_3 = 2 \\
    4x_1 - 5x_2 + 3x_3 = 4 
\end{cases}
\end{align*}

Answer. a) $(3t, t, -3t, t)$ b) $(1 - 2t, -t, t)$ \hfill \Box

(2) Find the inverses of the following matrices if they exist:

\begin{align*}
a) & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\
b) & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\
c) & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\end{align*}

Answer. a) $\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$ \\
b) $\begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$ \\
c) $\begin{bmatrix} -2/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & -2/3 \end{bmatrix}$ \hfill \Box

(3) Which of the following statements are true and which are false? Justify your answer.

(a) If $A^2 = A$, then $(I - A)^2 = I - A$, where $A$ is a square matrix.

Proof. True since
\begin{align*}
(I - A)^2 &= I^2 - IA - AI + A^2 \\
&= I - 2A + A^2 = I - 2A + A = I - A.
\end{align*}

\hfill \Box

\footnote{http://www.math.ualberta.ca/~xichen/math22514w/hw1sol.pdf}
(b) Every $5 \times 5$ skew symmetric matrix is singular.

**Proof.** True. Let $A$ be a $5 \times 5$ skew symmetric matrix. Then $A^T = -A$ and hence $\det(A^T) = \det(-A)$. And since $\det(A^T) = \det(A)$ and $\det(-A) = (-1)^5 \det(A)$, we obtain $\det(A) = -\det(A)$ and $\det(A) = 0$. So $A$ must be singular. □

(c) $(A + B)(A - B) = A^2 - B^2$ for all square matrices $A$ and $B$ of the same size.

**Solution.** False. We have $(A + B)(A - B) = A^2 - AB + BA - B^2$. So $(A + B)(A - B) = A^2 - B^2$ if and only if $AB = BA$. But $AB \neq BA$ in general, e.g.,

$$
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\neq
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
$$

□

(d) If $A^2 = I$, then $I - 2A$ is invertible, where $A$ is a square matrix.

**Proof.** True since

$$
4(A^2 - I) = -(I + 2A)(I - 2A) - 3I
$$

$$
\Rightarrow -\frac{1}{3}(I + 2A)(I - 2A) = I
$$

it follows that $I - 2A$ is invertible and

$$(I - 2A)^{-1} = -\frac{1}{3}(I + 2A).$$

□

(4) For which real values of $\lambda$ do the following vectors

$v_1 = (\lambda, 1, 1), \ v_2 = (1, \lambda, 1), \ v_3 = (1, 1, \lambda)$

form a linearly dependent set in $\mathbb{R}^3$?

**Solution.** $v_1, v_2, v_3$ are linearly dependent if and only if the matrix

$$
\begin{pmatrix}
\lambda & 1 & 1 \\
1 & \lambda & 1 \\
1 & 1 & \lambda
\end{pmatrix}
$$

is singular. □
is singular, i.e., when
\[ \det \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix} = \lambda^3 - 3\lambda + 2 = (\lambda + 2)(\lambda - 1)^2 = 0. \]

In conclusion, they are linearly dependent when \( \lambda = -2 \) or \( \lambda = 1 \). □

(5) Show that an \( n \times n \) matrix \( A \) is singular if and only if there exists a nonzero \( n \times n \) matrix \( B \) such that \( AB = 0 \).

Proof. Suppose that \( AB = 0 \) for some \( B \neq 0 \). Then \( A \) must be singular; otherwise, \( A \) is invertible and \( B = A^{-1}(AB) = 0 \), which is a contradiction.

Suppose that \( A \) is singular. Then there exists a nonzero column vector \( \mathbf{x} \) such that \( A\mathbf{x} = 0 \). Let
\[ B = \begin{bmatrix} \mathbf{x} & \mathbf{x} & \ldots & \mathbf{x} \\ \hline n \end{bmatrix}. \]

Then \( AB = 0 \). □

(6) Which of the following sets are vector spaces over \( \mathbb{R} \)? Justify your answer. Here vector addition and scalar multiplication are defined in the usual way unless stated otherwise.

(a) the set of odd functions \( f : \mathbb{R} \to \mathbb{R} \);

Proof. Yes. Let \( F(\mathbb{R}) \) be the vector space of all functions \( f : \mathbb{R} \to \mathbb{R} \). It suffices to show that \( O = \{ f \in F(\mathbb{R}) : f(-x) \equiv -f(x) \} \) is a subspace of \( F(\mathbb{R}) \).

Obviously, \( 0 \in O \). For \( f(x), g(x) \in O \), \( f(-x) = -f(x) \) and \( g(-x) = -g(x) \) for all \( x \). Therefore, \( f(-x) + cg(-x) = -(f(x) + cg(x)) \) for all \( x \) and \( c \in \mathbb{R} \) and hence \( f(x) + cg(x) \in O \). It follows that \( O \) is a subspace of \( F(\mathbb{R}) \) and hence a vector space. □

(b) the set of even functions \( f : \mathbb{R} \to \mathbb{R} \);

Proof. Yes. Let \( F(\mathbb{R}) \) be the vector space of all functions \( f : \mathbb{R} \to \mathbb{R} \). It suffices to show that \( E = \{ f \in F(\mathbb{R}) : f(-x) \equiv f(x) \} \) is a subspace of \( F(\mathbb{R}) \).

Obviously, \( 0 \in E \). For \( f(x), g(x) \in E \), \( f(-x) = f(x) \) and \( g(-x) = g(x) \) for all \( x \). Therefore, \( f(-x) + cg(-x) = f(x) + cg(x) \) for all \( x \) and \( c \in \mathbb{R} \) and hence \( f(x) + cg(x) \in E \).
It follows that $E$ is a subspace of $F(\mathbb{R})$ and hence a vector space.

c) $\mathbb{R}^2$ with vector addition $\oplus$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, 2(y_1 + y_2)).$$

**Solution.** This is not a vector space since law of associativity fails and also there is no zero vector. Suppose that there exists $o$ such that

$$u \oplus o = u$$

for all $u$. Let $o = (x_0, y_0)$. Then

$$(x, y) \oplus (x_0, y_0) = (x + x_0, 2(y + y_0)) = (x, y)$$

for all $(x, y)$. So $2y_0 = -y$ for all $y$, which is impossible. □

(7) Let $\mathbb{R}[x]$ be the vector space of all real polynomials in $x$. Determine whether the following sets are subspaces of $\mathbb{R}[x]$. Justify your answer.

(a) All polynomials $f(x)$ of degree $\leq 3$.

**Proof.** Yes, $W = \{ f(x) \in \mathbb{R}[x] : \deg f \leq 3 \}$ is a subspace of $\mathbb{R}[x]$. First, $0 \in W$ since $\deg 0 = -1$. Second, for all $f(x), g(x) \in W$, $\deg f \leq 3$ and $\deg g \leq 3$ and hence $\deg(f + cg) \leq 3$ for all $c \in \mathbb{R}$. Therefore, $f(x) + cg(x) \in W$ for all $f(x), g(x) \in W$ and $c \in \mathbb{R}$. So $W$ is a subspace. □

(b) All polynomials $f(x)$ satisfying $f(1) = f(2)$.

**Proof.** Yes, $W = \{ f(x) \in \mathbb{R}[x] : f(1) = f(2) \}$ is a subspace of $\mathbb{R}[x]$. First, $h(x) \equiv 0 \in W$ since $h(1) = h(2) = 0$. Second, for all $f(x), g(x) \in W$, $f(1) = f(2)$ and $g(1) = g(2)$ and hence $f(1) + cg(1) = f(2) + cg(2)$ for all $c \in \mathbb{R}$. Therefore, $f(x) + cg(x) \in W$ for all $f(x), g(x) \in W$ and $c \in \mathbb{R}$. So $W$ is a subspace. □

(c) All polynomials $f(x)$ satisfying $f(1) = 2$.

**Proof.** This is not a subspace of $\mathbb{R}[x]$ since it does not contain $0$. □

(d) All polynomials $f(x)$ satisfying $f''(1) = 0$. 

Proof. Yes, \( W = \{ f(x) \in \mathbb{R}[x] : f''(1) = 0 \} \) is a subspace of \( \mathbb{R}[x] \). First, \( h(x) \equiv 0 \in W \) since \( h''(x) \equiv 0 \). Second, for all \( f(x), g(x) \in W \), \( f''(1) = g''(1) = 0 \) and hence \((f + cg)''(1) = f''(1) + cg''(1) = 0\) for all \( c \in \mathbb{R} \). Therefore, \( f(x) + cg(x) \in W \) for all \( f(x), g(x) \in W \) and \( c \in \mathbb{R} \). So \( W \) is a subspace. \( \square \)

(8) Find the intersection \( V_1 \cap V_2 \) and the sum \( V_1 + V_2 \) of the subspaces \( V_1 \) and \( V_2 \) of \( \mathbb{R}^4 \) for
(a) \( V_1 = \{ x_1 + x_2 = 0 \} \) and \( V_2 = \{ x_3 + x_4 = 0 \} \);

Solution. The intersection is
\[
V_1 \cap V_2 = \{ x_1 + x_2 = x_3 + x_4 = 0 \}
= \left\{ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \right\}
= \operatorname{Nul} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.
\]

For \( V_1 + V_2 \), since
\[
(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp = \operatorname{Span}\{ (1, 1, 0, 0) \} \cap \operatorname{Span}\{ (0, 0, 1, 1) \}
= \{ \mathbf{x} : \mathbf{x} = (t_1, t_1, 0, 0) = (0, t_2, t_2) \}
= \{ (0, 0, 0, 0) \}
\]
it follows that
\[
V_1 + V_2 = \{ (0, 0, 0, 0) \}^\perp = \mathbb{R}^4.
\]
\( \square \)

(b) \( V_1 = \{ (t, t, -t, 0) \} \) and \( V_2 = \{ x_1 + x_2 = x_3 + x_4 = 0 \} \).

Solution. The intersection is
\[
V_1 \cap V_2 = \{ \mathbf{v} : \mathbf{v} \in V_1, \in V_2 \}
= \{ (t, t, -t, 0) : (t, t, -t, 0) \in V_2 \}
= \{ (t, t, -t, 0) : t + t = -t + 0 = 0 \} = \{ (0, 0, 0, 0) \}.
\]
We compute \( V_1 + V_2 \) through \((V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp\):
\[
V_1^\perp = \left( \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix} \right)^\perp
= \mathbf{v}^\perp.
\]
and
\[ V_2^\perp = \left( \text{Nul} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right)^\perp \]
\[ = \text{Row} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \text{Span}\{v_1, v_2\} \]

where \( v = (1, 1, -1, 0) \), \( v_1 = (1, 1, 0, 0) \) and \( v_2 = (0, 0, 1, 1) \).

Therefore,
\[ V_1^\perp \cap V_2^\perp = \{ a_1v_1 + a_2v_2 : \langle v, a_1v_1 + a_2v_2 \rangle = 0 \} \]
\[ = \{ a_1v_1 + a_2v_2 : 2a_1 - a_2 = 0 \} = \{ a_1(v_1 + 2v_2) \} \]
\[ = \text{Span}\{(1, 1, 2, 2)\} \]

and
\[ V_1 + V_2 = (1, 1, 2, 2)^\perp = \{ x_1 + x_2 + 2x_3 + 2x_4 = 0 \}. \]